THE QUEUEING SYSTEM M^X_2/M/n WITH HYSTERETIC CONTROL OF THE INPUT FLOW INTENSITY

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Abstract. We consider a multi-channel queueing system with unlimited queue and with exponentially distributed service time and the intervals between the arrival of customers batches, which uses a hysteretic control mechanism of the input flow intensity. The system receives two independent flows of customers, one of which is blocked in an overload mode. An algorithm for finding the stationary distribution of the number of customers and stationary characteristics (the mean queue length, the mean waiting time in the queue, the probability of customers loss) is proposed. The obtained results are verified with the help of a simulation model constructed with the assistance of GPSS World tools.

Keywords: multi-channel queueing system, batch arrival of customers, hysteretic control of the input flow intensity, stationary characteristics

Introduction

For the purpose of preventing overloads in the information and telecommunication systems simulated by means of queueing systems, hysteretic load control with several types of thresholds is used [1]. We can control both an input flow and its parameters, and service process.

According to [1], the queueing systems with hysteretic control may be adequate models for evaluating the quality of functioning of SIP servers under overloads. Detailed description of the use of such systems to the modeling of SIP servers operating under overload conditions is given in [1-3].

Because of the practical importance, a large number of publications is devoted to the study of queueing systems with hysteretic control. In particular, a sufficiently detailed overview of the results obtained in this direction can be found in [4-7]. Most studies examined a single-channel system with an arbitrary distribution of service time.

In this paper we consider a multi-channel queueing system M^X_2/M/n with the independent flows of two types of customers. In this system the control is applied only to the parameters of input flows. Switchings on the overload mode are carried
out not at the moments of the end of customers service, and at the moments of
arrival of customers in the system. It is more natural to practical applications.

When constructing a random process describing the number of customers in
the system, we share its states corresponding to the normal mode and the overload
mode. It allows us to write the system of equations for the stationary probabilities
and construct an algorithm to solve it. To find the stationary characteristics of
the queue, we apply the apparatus of generating functions. Using the same approach
in [8] the stationary characteristics of the system $M^X/M/1$ with hysteretic control
of service intensity are found.

1. Description of the system

Let us consider an $n$-channel $M^X_2/M/n$ queueing system that receives indepen-
dent flows of two types of customer batches. With probability $a_k$ (respectively $b_k$)
the number of customers in a batch of the first flow (respectively of the second
flow) is equal to $k$ ($k \geq 1$), and

$$\sum_{k=1}^{\infty} a_k = 1, \quad \sum_{k=1}^{\infty} b_k = 1; \quad a_{(s)} = \sum_{k=1}^{\infty} k^s a_k < \infty, \quad b_{(s)} = \sum_{k=1}^{\infty} k^s b_k < \infty, \quad s = 1, 2.$$ 

The time intervals between the time of arrival of customers batches of the flow
number $k$ are independent random variables distributed exponentially with
parameter $\lambda_k$ ($k = 1, 2$). In the total flow being a superposition of the first and
second type of flows, the time intervals between the time of arrival of customers
batches have exponential distribution with parameter $\lambda = \lambda_1 + \lambda_2$ [9, p. 83].
The service time of each customer is distributed exponentially with parameter $\mu$.
The first-in first-out (FIFO) service discipline is used.

A two-threshold hysteretic control mechanism of arriving load is realized in the
system. This mechanism operates as follows. We fix two numbers $h_1$ and $h_2$, for
which the following inequalities $n < h_1 < h_2$ hold. From the moment of arrival in
the system of the first customer and until the moment of presence of $h_2$ customers
for the first time, the system is functioning in a normal mode and customers of both
types are accepted for service. As soon as in the system there will be $h_2$ customers,
the system passes into an overload mode, accept of customers of the second type
stops and customers of the first type are accepted only. This mode continues until
the moment when the number of customers in the system is reduced to $(h_1 - 1)$.
At this moment the system passes into a normal mode of functioning and starts
over again to accept customers of both types. The normal mode operates up to
the moment where in the system there will be $h_1$ customers. Then the system goes
into overload mode, again accept of customers of the second type stops and so on.
We assume that the following inequalities $h_1 - n \geq 2$ and $h_2 - h_1 \geq 2$ hold. These assumptions are introduced only in order not to consider cases for which the formulas are different from those shown here, and in no way detract from the generality of the obtained results.

2. Stationary distribution of the number of customers

Let $\alpha = \lambda / \mu$, $\alpha_i = \lambda_i / \mu$ ($i = 1, 2$), $\gamma = \alpha / n$, $\gamma_i = \alpha_i / n$, ($i = 1, 2$), then $\rho = \gamma_1 a_1 + \gamma_2 b_1$ and $\rho_i = \gamma_i a_i(1)$ are coefficients of system loading in a normal mode and in an overload mode respectively.

Let us enumerate the system’s states as follows (see Fig. 1): $s_0$ corresponds to the empty system; $s_i$ ($1 \leq i \leq h_2 - 1$), to the system with $i$ customers that operates in the normal mode; $x_i$ ($i \geq h_1$), to the system with $i$ customers that operates in the overload mode.

![Fig. 1. Diagram of the system’s states in the case $a_1 = h_1 = 1$](image)

Let $p_i(t)$ and $q_i(t)$ be the probability of the fact that at instant $t$ the system is in state $s_i$ and $x_i$ respectively. Assuming that the limits $p_i = \lim_{t \to \infty} p_i(t)$ ($0 \leq i \leq h_2 - 1$) and $q_i = \lim_{t \to \infty} q_i(t)$ ($i \geq h_1$) exist (the ergodicity conditions to be obtained below), we can represent the system of equations for stationary probabilities $p_i$ and $q_i$ in the form

$$-\lambda p_0 + \mu p_1 = 0;$$

$$-(\lambda + i\mu) p_i + \sum_{k=0}^{i-1} p_k(\lambda a_{i-k} + \lambda b_{i-k}) + (i+1)\mu p_{i+1} = 0 \quad (1 \leq i \leq n - 1);$$

$$-(\lambda + n\mu) p_i + \sum_{k=0}^{i-1} p_k(\lambda a_{i-k} + \lambda b_{i-k}) + n\mu p_{i+1} = 0 \quad (n \leq i \leq h_2 - 2; \ i \neq h_1 - 1);$$

$$-(\lambda + n\mu) p_{h_1 - 1} + \sum_{k=0}^{h_2 - 2} p_k(\lambda a_{h_1 - 1-k} + \lambda b_{h_1 - 1-k}) + n\mu (p_{h_1} + q_{h_1}) = 0;$$

$$-(\lambda + n\mu) p_{h_2 - 1} + \sum_{k=0}^{h_2 - 2} p_k(\lambda a_{h_2 - 1-k} + \lambda b_{h_2 - 1-k}) = 0;$$
\[-(\lambda_1 + n\mu)q_h + n\mu q_{h+1} = 0;\]
\[-(\lambda_1 + n\mu)q_i + \lambda_1 \sum_{k=h_1}^{h+1-1} q_k a_{i-k} + n\mu q_{i+1} = 0 \quad (h_1 + 1 \leq i \leq h_2 - 1);\]
\[-(\lambda_1 + n\mu)q_{h_2} + \lambda_1 \sum_{k=h_1}^{h_2-1} q_k a_{b_{h-2-k}} + \sum_{k=0}^{h_2-1} p_k (\lambda_2 a_{b_{h-2-k}} + \lambda_2 B_{b_{h_2-1-k}}) + n\mu q_{h_2+1} = 0;\] (2)
\[-(\lambda_1 + n\mu)q_{i} + \lambda_1 \left( \sum_{k=h_1}^{i-1} q_k + \sum_{k=0}^{h_2-1} p_k \right) a_{i-k} + n\mu q_{i+1} = 0 \quad (i \geq h_2);\]
\[\sum_{k=0}^{h_2-1} p_k + \sum_{k=h_1}^{h_2} q_k = 1.\] (3)

Here
\[B_i = 1 - \sum_{k=i}^{\infty} b_k \quad (i \geq 0), \quad B_0 = 1.\]

Let us introduce the notations
\[A_i = 1 - \sum_{k=1}^{i} a_k \quad (i \geq 0), \quad A_0 = 1; \quad \tilde{p}_k = p_k / p_0 \quad (0 \leq k \leq h_2 - 1);\]
\[\bar{q}_k = q_k / p_0 \quad (k \geq h_1); \quad P_i = \sum_{k=0}^{i} \tilde{p}_k; \quad L_n = \sum_{k=1}^{n} k \tilde{p}_k; \quad Q_{b_{h_2-1}} = \sum_{k=h_1}^{b_{h_2-1}} \bar{q}_k.\] (4)

**Theorem 1.** When \(a_{(1)} < \infty, \quad b_{(1)} < \infty\) and \(\rho_1 < 1\), the stationary probabilities \(p_i = p_0 \tilde{p}_i \quad (0 \leq i \leq h_2 - 1)\) and \(q_i = p_0 \bar{q}_i \quad (i \geq h_1)\) exist and are determined from the recurrence relations
\[p_0 = \frac{n - \alpha(a_{(1)})}{nP_n - L_n + \alpha_2 \sum_{r=0}^{h_2-1} \sum_{k=0}^{i} \tilde{p}_k B_{i-k}};\] (5)
\[\tilde{p}_{i+1} = \frac{1}{i+1} \sum_{k=0}^{i} \tilde{p}_k (\alpha_1 A_{i-k} + \alpha_2 B_{i-k}) \quad (0 \leq i \leq n-1);\] (6)
\[\bar{p}_{i+1} = \frac{1}{n} \sum_{k=0}^{i} \tilde{p}_k (\alpha_1 A_{i-k} + \alpha_2 B_{i-k}) \quad (n \leq i \leq h_1 - 2);\] (7)
\[\tilde{p}_{h_2} = (1 + \gamma) \tilde{p}_i - \sum_{k=0}^{i} \tilde{p}_k (\gamma_1 a_{i-k} + \gamma_2 b_{i-k}) \quad (h_1 \leq i \leq h_2 - 2);\] (8)
\[\tilde{p}_{h_2-1} = \frac{1}{1 + \gamma} \sum_{k=0}^{h_2-2} \tilde{p}_k (\gamma_1 a_{h_2-1-k} + \gamma_2 b_{h_2-1-k});\] (9)
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\[
\tilde{q}_{hi} = \sum_{k=0}^{h_k-2} \tilde{p}_k (\gamma_1 A_{h_2-i-k} + \gamma_2 B_{h_2-i-k}); \tag{10}
\]

\[
\check{q}_{i+1} = \check{q}_{hi} + \gamma_1 \sum_{k=0}^{h_k-1} \check{q}_k A_{-k} \quad (h_1 \leq i \leq h_2 - 1); \tag{11}
\]

\[
\check{q}_{i+1} = \gamma_1 \left( \sum_{k=0}^{h_k-1} \check{q}_k + \sum_{k=0}^{h_k-1} \tilde{p}_k \right) A_{-k} \quad (i \geq h_2); \tag{12}
\]

and \( \tilde{p}_{h_1} \) is determined from the equation

\[
\frac{1}{1 + \gamma} \sum_{k=0}^{h_k-2} \tilde{p}_k (\gamma_1 a_{h_2-1-k} + \gamma_2 b_{h_2-1-k}) = (1 + \gamma) \tilde{p}_{h_2-1} - \sum_{k=0}^{h_k-3} \tilde{p}_k (\gamma_1 a_{h_2-2-k} + \gamma_2 b_{h_2-2-k}). \tag{13}
\]

**Proof.** The successive summation of the first \( i \) \((0 \leq i \leq h_1 - 2)\) equations of system (1) yields relations (6) and (7). We continue this process for \( h_1 - 1 \leq i \leq h_2 - 2 \), to obtain the equalities

\[
n\mu (\tilde{p}_{i+1} + \tilde{q}_{h_1}) = \sum_{k=0}^{i} \tilde{p}_k (\lambda_1 A_{-k} + \lambda_2 B_{-k}) \quad (h_1 - 1 \leq i \leq h_2 - 2), \tag{14}
\]

which yield relations (8). Equality (9) follows from the last equation of system (1). We equate the expressions for \( \tilde{p}_{h_2-1} \) that follow from (8) and (9) to obtain equation (13). In order to find \( \tilde{p}_{h_1} \) it is necessary to express all \( \tilde{p}_k \) for \( h_1 + 1 \leq k \leq h_2 - 2 \) in this equation by \( \tilde{p}_{h_1} \).

The summation of the last equality from (14) and last equation of system (1) yields relation (10). Equalities (11) are obtained as a result of the successive summation of equations (2) for \( h_1 \leq i \leq h_2 - 1 \).

Multiplying the last equality from (11) by \( n\mu \) and successive summing of the result with equation (2) for \( i \geq h_2 \), taking into account (10), we obtain relations (12).

Divide both sides of (14) by \( n\mu \) and add the result with (11) for \( h_1 \leq i \leq h_2 - 1 \). Taking into account the first of the relations (14), we obtain the equalities

\[
\tilde{p}_{i+1} + \tilde{q}_{i+1} = \gamma_1 \left( \sum_{k=0}^{i} \tilde{p}_k A_{-k} + \sum_{k=0}^{i} \check{q}_k A_{-k} \right) + \gamma_2 \sum_{k=0}^{i} \tilde{p}_k B_{-k} \quad (h_1 - 1 \leq i \leq h_2 - 2). \tag{15}
\]
In order to find $p_0$, first with the help of (6), (7), (15), (11) (taking into account (10)) and (12) we obtain the equality
\[
\sum_{k=1}^{n} kp_k + n \left( \sum_{k=n+1}^{h_0-1} p_k + \sum_{k=h_1}^{\infty} q_k \right) = 
\]
\[
= \alpha_1 \left( \sum_{i=0}^{h_0-1} \sum_{k=0}^{\infty} p_k A_{i-k} + \sum_{i=h_2}^{h_0-1} \sum_{k=0}^{\infty} p_k B_{i-k} + \alpha_2 \sum_{i=0}^{h_0-1} \sum_{k=0}^{\infty} p_k A_{i-k} \right). 
\]
Then taking into account the definitions of $A_i$, $B_i$, $a_{(1)}$, $b_{(1)}$ and normalization condition (3) we calculate the sums
\[
\left( \sum_{i=0}^{h_0-1} \sum_{k=0}^{\infty} p_k A_{i-k} + \sum_{i=h_2}^{h_0-1} \sum_{k=0}^{\infty} p_k A_{i-k} \right) = \alpha_1 \sum_{k=0}^{h_0-1} p_k; 
\]
\[
\sum_{i=h_1}^{\infty} \sum_{k=0}^{\infty} q_k A_{i-k} = \sum_{i=1}^{h_1} \sum_{k=0}^{\infty} A_k = a_{(1)} \left( 1 - \sum_{k=0}^{h_1} p_k \right). 
\]
In view of (17) we rewrite (16) as
\[
\alpha_1 a_{(1)} + \alpha_2 \sum_{k=0}^{h_0-1} p_k B_{i-k} = \sum_{k=1}^{n} kp_k + n \left( \sum_{k=n+1}^{h_0-1} p_k + 1 - \sum_{k=0}^{h_0-1} p_k \right) = \sum_{k=1}^{n} kp_k + n \left( 1 - \sum_{k=0}^{h_1} p_k \right). 
\]
Taking into account the definitions of $P_s$ and $L_n$ (see (4)), we transform the obtained equation to the form
\[
\alpha_1 a_{(1)} + \alpha_2 p_{(1)} \sum_{i=0}^{h_0-1} \sum_{k=0}^{h_0-1} \tilde{p}_k B_{i-k} = n + p_0 (L_n - n p_n). 
\]
Relation (18) implies formula (5), which yields positive values of $p_0$ only under the condition $\alpha_1 a_{(1)} < n$, i.e., when $\rho_1 < 1$.

Thus, when the conditions $a_{(1)} < \infty$, $b_{(1)} < \infty$ and $\rho_1 < 1$ are fulfilled, the system of equilibrium equations (1) and (2) has a nontrivial solution such that equality (3) is valid. Let $\{\xi(t), t \geq 0\}$ be a random process with discrete states $s_i$ ($0 \leq i \leq h_2 - 1$) and $x_i$ ($i \geq h_1$). Since all the states of this irreducible Markovian process communicate, the regularity criterion [9, p. 40] implies the regularity of this process, and it follows from the Foster ergodic theorem [9, p. 48] that the conditions of Theorem 1 are sufficient for the existence of stationary probabilities $p_i$ ($0 \leq i \leq h_2 - 1$) and $q_i$ ($i \geq h_1$). The theorem is proved.
3. Generating function and its derivative

Let \( N(F(z)) \) and \( D(F(z)) \) denote the numerator and denominator, respectively, in the expression for certain function \( F(z) \). Consider the generating functions

\[
P(z) = \sum_{i=0}^{h_1-1} p_i z^i; \quad Q(z) = \sum_{i=h_1}^{\infty} q_i z^i; \quad A(z) = \sum_{i=1}^{\infty} a_i z^i.
\]

**Theorem 2.** Generating function \( Q(z) \) is determined in the form

\[
Q(z) = \frac{N(Q(z))}{D(Q(z))}; \quad D(Q(z)) = n\mu(1-z) - \lambda_i z(1-A(z));
\]

\[
N(Q(z)) = n\mu q_{h_1} z^{h_1} - \lambda_i z A(z) P(z) + \lambda_i \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-i-k} a_i z^{k+i+1} - \lambda_i z^{h_2+1} \sum_{k=0}^{h_1-1} p_k B_{h_2-i-k}.
\]  

(19)

**Proof.** The multiplication of the \( i \)th equation of system (2) by \( z^i \) \( (i \geq h_1) \) and summation yield

\[
-(\lambda_i + n\mu)Q(z) + n\mu \frac{Q(z)}{z} - n\mu q_{h_1} z^{h_1-1} + \lambda_i \sum_{i=h_1+1}^{\infty} z^i \sum_{k=h_1}^{i-1} q_k a_{i-k} +
\]

\[
+ \lambda_i \sum_{i=h_2}^{\infty} z^i \sum_{k=0}^{h_1-1} p_k a_{i-k} + \lambda_i z^{h_2} \sum_{k=0}^{h_1-1} p_k B_{h_2-i-k} =
\]

\[
= -(\lambda_i + n\mu)Q(z) + n\mu \frac{Q(z)}{z} - n\mu q_{h_1} z^{h_1-1} + \lambda_i A(z)(P(z) + Q(z)) -
\]

\[
-\lambda_i \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-i-k} a_i z^{k+i} + \lambda_i z^{h_2} \sum_{k=0}^{h_1-1} p_k B_{h_2-i-k} = 0.
\]

Solving the last equation, we find \( Q(z) \) in form (19). The theorem is proved.

Let us simplify the calculation of the derivative of function \( Q(z) \).

**Lemma 1.** For the derivative of function \( Q(z) \) for \( z = 1 \) the following formula

\[
Q'(1) = \frac{N''(Q'(1))}{D''(Q'(1))} = \frac{(n\mu - \lambda_i a_{(1)}) N''(Q'(1)) + \lambda_i (a_{(1)} + a_{(2)}) N'(Q'(1))}{2(n\mu - \lambda_i a_{(1)})^2}
\]  

(20)

is valid where
\[ N'(Q)(1) = n \mu h_1 q_{h_1} - \lambda_1 \left( a_{(1)} \sum_{k=0}^{h_2-1} p_k - \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} (k+i+1) a_i \right) - \lambda_1 \sum_{k=0}^{h_2-1} (k+1) p_k - \lambda_2 (h_2+1) \sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}, \]
\[ N''(Q)(1) = n \mu h_1 (h_1-1) q_{h_1} - \lambda_1 \left( 2 a_{(1)} \sum_{k=1}^{h_2-1} k p_k + \sum_{k=1}^{h_2-1} k(k+1) p_k + (a_{(1)} + a_{(2)}) \sum_{k=0}^{h_2-1} p_k \right) + \lambda_1 \sum_{k=0}^{h_2-2} p_k \sum_{i=1}^{h_2-1-k} (k+i+1) a_i - \lambda_2 h_2 (h_2+1) \sum_{k=0}^{h_2-1} p_k B_{h_2-1-k}. \]

**Proof.** We calculate the derivative \( Q'(1) \) taking into account the equalities
\[ A'(1) = \sum_{k=1}^{\infty} k a_k = a_{(1)}; \quad A''(1) = \sum_{k=2}^{\infty} k(k-1) a_k = a_{(2)} - a_{(1)}. \]
Since \( A(1) = \sum_{k=1}^{\infty} a_k = 1 \), as a result of addition of the equations (1), we see that \( N(Q)(1) = 0 \).

So, function \( Q(z) \) specified by formulas (19) satisfies the equalities
\[ N(Q)(1) = D(Q)(1) = N(Q')(1) = D(Q')(1) = N''(Q')(1) = D'(Q')(1) = 0; \quad D''(Q')(1) \neq 0. \] (21)

The first part of formulas (20) follows from (21). Since
\[ N(Q')(z) = N'(Q)(z) \cdot D(Q)(z) - N(Q)(z) \cdot D'(Q)(z); \]
\[ N'(Q')(z) = N''(Q)(z) \cdot D(Q)(z) - N(Q)(z) \cdot D''(Q)(z); \]
\[ N''(Q')(z) = N'''(Q)(z) \cdot D(Q)(z) + N''(Q)(z) \cdot D'(Q)(z) - N'''(Q)(z) \cdot D''(Q)(z) - N'(Q)(z) \cdot D'''(Q)(z) - N(Q)(z) \cdot D''''(Q)(z), \]
taking into account (21) we obtain the equality
\[ N'''(Q')(1) = N''(Q)(1) \cdot D'(Q)(1) - N'(Q)(1) \cdot D''(Q)(1) = -(n \mu - \lambda_1 a_{(1)}) N''(Q)(1) - \lambda_1 (a_{(1)} + a_{(2)}) N'(Q)(1). \]

Thus, the lemma is proved.

**4. Stationary characteristics**

Using the stationary distribution of the number of customers and the derivative \( Q'(1) \) we can obtain simple formulas for the main stationary characteristics of the system.
Given that
\[ Q'(1) = \sum_{k=0}^{\infty} kq_k, \]
the formulas for the stationary mean queue length in the system and the stationary mean number of customers in the system
\[
E(Q) = \sum_{k=n+1}^{h_2-1} (k-n)p_k + \sum_{k=h_1}^{\infty} (k-n)q_k, \quad E(N) = \sum_{k=1}^{h_2-1} kp_k + \sum_{k=h_1}^{\infty} kq_k,
\]
we reduce to the form
\[
E(Q) = Q'(1) + \sum_{k=n+1}^{h_2-1} kp_k - n\left(1 - \sum_{k=0}^{n} p_k\right), \quad E(N) = \sum_{k=1}^{h_2-1} kp_k + Q'(1).
\]

The formula for the stationary mean number of served customers per unit of time
\[
N_{sv} = \mu \left( \sum_{k=1}^{n-1} kp_k + n \left( \sum_{k=n}^{h_2-1} p_k + \sum_{k=h_1}^{\infty} q_k \right) \right).
\]
after using of the normalization condition (3) takes the form
\[
N_{sv} = \mu \left( n(1-p_0) - \sum_{k=1}^{n-1} (n-k)p_k \right). \tag{22}
\]
On the other hand, the served load is determined by the formula
\[
N_{sv} = \lambda_1 a_1(1) + \lambda_2 \sum_{i=0}^{h_2-1} \sum_{k=0}^{i} p_k B_{i-k}. \tag{23}
\]
Equating the expressions (22) and (23), we again arrive at the balance equation (18), by which we determine the stationary probability \( p_0 \) of free system.

The stationary mean waiting time in the queue equals
\[
E(W) = \frac{E(Q)}{N_{sv}},
\]
and the stationary loss probability of customers \( \pi \) and the stationary loss probability of customers of the second type \( \pi_2 \) have the form
\[
\pi = 1 - \frac{N_{sv}}{\lambda_1 a_1(1) + \lambda_2 b_1(1)}, \quad \pi_2 = 1 - \frac{1}{b_1(1) \sum_{i=0}^{h_2-1} \sum_{k=0}^{i} p_k B_{i-k}}.
\]
Recall that customers of first type are not lost. To find $\pi_2$ we have used relation (23) and the equality $N_{sv} = \lambda_1 a_{(1)} + \lambda_2 b_{(1)} (1 - \pi_2)$.

5. The case of ordinary input flow

Since the ordinary flow of customers following equalities hold

$$a_1 = b_1 = 1; \quad a_k = b_k = 0 \quad (k \geq 2); \quad A_k = B_k = 0 \quad (k \geq 1),$$

the recurrence relations (6)-(12) are simplified and allow us to obtain expressions for $\tilde{p}_k$ ($1 \leq k \leq h_2 - 1$) and $\tilde{q}_k$ ($k \geq h_1$) in explicit form

$$\tilde{p}_k = \frac{\alpha^k}{k!} (1 \leq k \leq n); \quad \tilde{q}_k = \frac{\alpha^k}{n! n^{n-k}} (n + 1 \leq k \leq h_1 - 1),$$

$$\tilde{p}_k = \sum_{h_2-h_1}^{h_1-1} \gamma^j \tilde{p}_{h_1-j} \quad (h_1 \leq k \leq h_2 - 1); \quad \tilde{q}_k = \frac{\gamma (1 - \gamma_1^{k-h_2+1})}{1 - \gamma_1} \tilde{p}_{h_2-1} \quad (h_1 \leq k \leq h_2);$$

$$\tilde{q}_k = \frac{\gamma \gamma_1^{k-h_2} (1 - \gamma_1^{h_2-h_1})}{1 - \gamma_1} \tilde{p}_{h_2-1} \quad (k \geq h_2 + 1).$$

Taking into account that

$$\sum_{k=h_2}^{\infty} \tilde{q}_k = \frac{\tilde{q}_{h_2}}{1 - \gamma_1},$$

and using the normalization condition (3) we obtain the formula for $p_0$

$$p_0 = \frac{1}{p_{h_2-1} + Q_{h_2-1} + \frac{\tilde{q}_{h_2}}{1 - \gamma_1}}.$$ 

Condition for the existence of stationary probabilities in the case of the ordinary input flow takes the form $\gamma_1 < 1$.

Since

$$\frac{1}{\tilde{q}_{h_2}} \sum_{k=h_2}^{\infty} k \tilde{q}_k = \sum_{k=h_2}^{\infty} (k - h_2) \gamma_1^{k-h_2} + h_2 \sum_{k=h_2}^{\infty} \gamma_1^{k-h_2} = h_2 \sum_{k=0}^{\infty} \frac{\gamma_1^k}{1 - \gamma_1} + \frac{h_2}{1 - \gamma_1} = \frac{\gamma_1 + h_2 (1 - \gamma_1)}{(1 - \gamma_1)^2},$$
for the stationary mean queue length in the system we obtain the expression
\[
E(Q) = \sum_{k=n+1}^{b_2-1} (k-n) p_k + \sum_{k=h_1}^{b_2-1} (k-n) q_k + \frac{\gamma_1 + (1-\gamma_1)(h_2-n)}{(1-\gamma_1)^2} q_{h_2}.
\]

6. Examples of calculation of the stationary characteristics

Let us present the results of calculations of stationary characteristics of the system, performed using the obtained analytical relations.

Calculations are performed for the three-channel queueing system \(n = 3\) for the values of thresholds \(h_1 = 7,\ h_2 = 12\) and following values of parameters characterizing the intensity of service and the intensity of customers flow: \(\mu = 1;\ \lambda = 4,5;\ a_1 = 0,5,\ a_2 = 0,3,\ a_3 = 0,2;\ b_1 = 0,2,\ b_2 = 0,3,\ b_3 = 0,5;\ a_k = b_k = 0\ (k \geq 4).\) So, customers arrive in batches numbering from one to three and with different probabilities distributions of the number of customers in the batch for each flow. The mean number of customers in the batch for the first type and second type of flows equals \(a_{(1)} = 1,7\) and \(b_{(1)} = 2,3\) respectively.

For a fixed value \(\lambda = 4,5\) of the parameter of the exponential distribution of time intervals between the moments of arrival of customers batches of total flow we consider the following pairs of values of the parameters \(\lambda_1, \lambda_2\) for flows of the first and the second type:
1) \(\lambda_1 = 0,5;\ \lambda_2 = 4;\ \bar{N}_\lambda = 10,05;\)
2) \(\lambda_1 = 1;\ \lambda_2 = 3,5;\ \bar{N}_\lambda = 9,75;\)
3) \(\lambda_1 = 1,5;\ \lambda_2 = 3;\ \bar{N}_\lambda = 9,45;\)

Here \(\bar{N}_\lambda = \lambda_1 a_{(1)} + \lambda_2 b_{(1)}\) is an intensity of the total flow of customers during the normal operating mode of the system.

The stationary probabilities \(p_k\) \((k \geq 0)\) of availability in the system of \(k\) customers computed for each pair 1)–3) of values \(\lambda_1\) and \(\lambda_2\) are shown in Table 1. Bold are maximum values of \(p_k\) which are reached for \(k = 9,\ k = 11\) and \(k = 12\) respectively.

Table 2 shows the values of the stationary characteristics computed for the same pairs of values \(\lambda_1\) and \(\lambda_2\). For comparison, the values of these parameters obtained by simulation system GPSS World \([10, 11]\) for the value of the simulation time \(t = 3 \times 10^5\) also are presented in Table 2.

Analysis of the obtained results shows that, despite the close values of the total flow intensity \(\bar{N}_\lambda\) for cases 1)–3), the characteristics of the queue of the system for the considered input data deteriorate significantly with increasing flow intensity \(\lambda_1 a_{(1)}\) of the first type customers.
Table 3 shows the values of the stationary mean queue length $E(Q)$ calculated using the GPSS World for various laws of distribution of service time: exponential distribution with parameter $\mu = 1$, uniform distribution on the interval $[0, 2]$, uniform distribution on the interval $[0.5, 1.5]$ and deterministic value of 1.

### Table 1

**Stationary distributions of the number of customers in the system**

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$p_0$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
<th>$p_5$</th>
<th>$p_6$</th>
<th>$p_7$</th>
<th>$p_8$</th>
<th>$p_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4</td>
<td>0.0002</td>
<td>0.0010</td>
<td>0.0026</td>
<td>0.0052</td>
<td>0.0115</td>
<td>0.0251</td>
<td>0.0545</td>
<td>0.1186</td>
<td>0.1394</td>
<td><strong>0.1504</strong></td>
</tr>
<tr>
<td>1</td>
<td>3.5</td>
<td>0.0002</td>
<td>0.0007</td>
<td>0.0019</td>
<td>0.0037</td>
<td>0.0080</td>
<td>0.0173</td>
<td>0.0371</td>
<td>0.0799</td>
<td>0.1032</td>
<td>0.1202</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
<td>0.0001</td>
<td>0.0003</td>
<td>0.0008</td>
<td>0.0015</td>
<td>0.0033</td>
<td>0.0070</td>
<td>0.0148</td>
<td>0.0315</td>
<td>0.0441</td>
<td>0.0557</td>
</tr>
</tbody>
</table>

### Table 2

**Stationary system characteristics for different pairs of values $\lambda_1$ and $\lambda_2$**

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$E(Q)$ (GPSS)</th>
<th>$E(Q)$ (W)</th>
<th>$E(W)$ (GPSS)</th>
<th>$\pi_1$ (GPSS)</th>
<th>$\pi_2$ (GPSS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4</td>
<td>6.324</td>
<td>6.454</td>
<td>2.112</td>
<td>2.156</td>
<td>0.702</td>
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<tr>
<td>1</td>
<td>3.5</td>
<td>8.103</td>
<td>8.184</td>
<td>2.704</td>
<td>2.739</td>
<td>0.693</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
<td>14.482</td>
<td>14.428</td>
<td>4.830</td>
<td>4.814</td>
<td>0.683</td>
</tr>
</tbody>
</table>

### Table 3

Values of the stationary mean queue length $E(Q)$ for different pairs of values $\lambda_1$ and $\lambda_2$ calculated using the GPSS World for various laws of distribution of service time: 1 - exponential distribution with parameter $\mu = 1$; 2 - uniform distribution on the interval $[0, 2]$; 3 - uniform distribution on the interval $[0.5, 1.5]$; 4 - deterministic value of 1

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>4</td>
<td>6.454</td>
<td>6.307</td>
<td>6.234</td>
<td>4.429</td>
</tr>
<tr>
<td>1</td>
<td>3.5</td>
<td>8.184</td>
<td>7.438</td>
<td>7.725</td>
<td>6.990</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
<td>14.428</td>
<td>12.754</td>
<td>11.958</td>
<td>11.633</td>
</tr>
</tbody>
</table>
Conclusions

Thus, in this paper we have presented a simple algorithm for finding of the stationary distribution of the number of customers and stationary characteristics of a queueing system $M^X_2/M/n$. The system receives two independent nonordinary flows of customers and uses a hysteresis strategy of control of the input flow intensity. The results of test calculations are given. The obtained relations can be used to solve optimization problems associated with a hysteretic control mechanism.

Analyzing the data from Table 3, we conclude that the formulas for the case of exponential service time distribution obtained in this paper can be used for evaluative analysis of systems of type $M^X_2/G/n$ with hysteretic control of the input flow intensity.

References


[8] Zhernovyi Yu.V., Stationary characteristics of $M^X/G/1$ systems with hysteretic switching of the service intensity, Journal of Communications Technology and Electronics 2013, 58, 6, 613-627.

