Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

CALCULATING STEADY-STATE PROBABILITIES OF CLOSED QUEUEING SYSTEMS USING HYPEREXPONENTIAL APPROXIMATION

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Abstract. In this paper we propose a method for calculating steady-state probability distributions of the closed queueing systems with exponential distribution of customer generation time and arbitrary distribution of service times. The approach based on the use of fictitious phases and hyperexponential approximations with parameters of the paradoxical and complex type by method of moments. We consider queueing systems with the number of channels $n = 1$, 2 and 3. The obtained numerical results are verified using simulation models.

1. Introduction

The purpose of this work is the analysis of a model of a closed queueing system that is employed, in particular, in the theory of communication networks and integral queueing networks [1, 3–5]. A closed system is also known as a system with a finite number of sources or the Engset system.

We assume that customers from m identical sources are fed to a queueing system. Each source can generate only one customer, and the next customer is not sent if the previous customer is not processed. Time interval from the moment at which the customer is returned to the source to the moment of the arrival of this customer to the system is the customer generation time. We assume that the customer generation time is distributed in accordance with the exponential law with parameter λ. Intensity of the input flow of customers of a closed system depends on the number of customers in the system $\xi(t)$ at moment t and is represented as $\lambda(m - \xi(t))$. Such a flow is classified as the Poisson flow of the second kind (see [2]).

The method of potentials was used in [8] to construct an algorithm that makes it possible to determine the steady-state distribution of the number of customers for a single-channel closed system with an arbitrary distribution of service times. This method is not suitable for multichannel queueing systems.

Article [6] shows that the use of hyperexponential approximation (H_r) makes it possible to determine with high accuracy the steady-state probabilities of non-Markovian single-channel queuing systems. These probabilities are determined

1991 *Mathematics Subject Classification.* 60G10, 60J28, 60K25, 93B40.

Key words and phrases. Closed queueing system, arbitrary distribution of service times, hyperexponential approximation, fictitious phases, method of moments.

using solutions of a system of linear algebraic equations obtained by the method of fictitious phases. To find parameters of the H_r-approximation of a certain distribution it is sufficient to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, roots of this system are complex-valued or paradoxical (i.e., negative or with probabilities that exceed the boundaries of the interval [0, 1]) but in most cases as a result of summation of probabilities of microstates, their complex-valued and paradoxical parts are annihilated.

The purpose of the paper is to use of the hyperexponential approximation method for calculating steady-state probabilities of the closed queueing systems with exponential distribution of customer generation time and arbitrary distribution of service times. We consider queueing systems with the number of channels $n = 1$, 2 and 3. The obtained numerical results are verified using simulation models. The results for a single-channel system can be checked using the method of potentials. We also indicate ways to evaluate the accuracy of approach the obtained steady-state distribution to the true distribution without the need to use simulation models.

2. Equations for steady-state probabilities of the single-channel closed system with H_r-distribution of service times

The hyperexponential distribution of order r is a phase-type distribution and provides for choosing one of r alternative phases by a random process. With probability β_i, the process is at the ith phase and is in it during an exponentially distributed time with a parameter μ_i.

Let us suppose that the customer generation time is exponentially distributed with a parameter λ, and the service times of each customer are independent random variables distributed according to the hyperexponential law H_r $(r \geq 2)$ with probabilities β_i and parameters μ_i $(1 \leq i \leq r)$.

Let us enumerate the single-channel system's states as follows: x_0 corresponds to the empty system; $x_{k(i)}$ is the state, when there are k customers in the system $(1 \leq k \leq m)$, the service time is in the phase i $(1 \leq i \leq r)$. We denote by p_0 and $p_{k(i)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate p_0 and $p_{k(i)}$ we obtain the system of linear equations:

$$\begin{aligned}
&-m\lambda p_0 + \sum_{i=1}^{r} \mu_i p_{1(i)} = 0;\\
&-((m-1)\lambda + \mu_i)\, p_{1(i)} + m\lambda\beta_i p_0 + \beta_i \sum_{j=1}^{r} \mu_j p_{2(j)} = 0, \quad 1 \leq i \leq r;\\
&-((m-k)\lambda + \mu_i)\, p_{k(i)} + (m-k+1)\lambda p_{k-1(i)} + \beta_i \sum_{u=1}^{r} \mu_u p_{k+1(u)} = 0,\\
&\qquad\qquad\qquad\qquad\qquad\qquad 2 \leq k \leq m-1, \quad 1 \leq i \leq r;\\
&-\mu_i p_{m(i)} + \lambda p_{m-1(i)} = 0, \quad 1 \leq i \leq r; \qquad p_0 + \sum_{k=1}^{m} \sum_{i=1}^{r} p_{k(i)} = 1.
\end{aligned} \tag{2.1}$$

Solving the system (2.1), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$p_k = \sum_{i=1}^{r} p_{k(i)}, \quad 1 \le k \le m. \tag{2.2}$$

3. Equations for steady-state probabilities of the double-channel closed system with H_r-distribution of service times

Let us enumerate the double-channel system's states as follows: x_0 corresponds to the empty system; $x_{k(ij)}$ is the state, when there are k customers in the system $(1 \le k \le m)$, the service time is in the phase i $(1 \le i \le r)$, and in the phase j $(i \le j \le r)$ for each of the two channels respectively. For the case when $k = 1$, we assume that $j = 0$. We denote by p_0 and $p_{k(ij)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate p_0 and $p_{k(ij)}$ we obtain the system of linear equations:

$$\begin{aligned}
&-m\lambda p_0 + \sum_{u=1}^{r} \mu_u p_{1(u0)} = 0;\\
&-\left((m-1)\lambda + \mu_i\right) p_{1(i0)} + m\lambda\beta_i p_0 + 2\mu_i p_{2(ii)} + \sum_{u=1}^{i-1} \mu_u p_{2(ui)} +\\
&\qquad\qquad + \sum_{u=i+1}^{r} \mu_u p_{2(iu)} = 0, \quad 1 \le i \le r;\\
&-\left((m-2)\lambda + 2\mu_i\right) p_{2(ii)} + (m-1)\lambda\beta_i p_{1(i0)} + 2\mu_i\beta_i p_{3(ii)} +\\
&\quad + \beta_i \left(\sum_{u=1}^{i-1} \mu_u p_{3(ui)} + \sum_{u=i+1}^{r} \mu_u p_{3(iu)} \right) = 0, \quad 1 \le i \le r;\\
&-\left((m-2)\lambda + \mu_i + \mu_j\right) p_{2(ij)} + (m-1)\lambda\beta_j p_{1(i0)} + (m-1)\lambda\beta_i p_{1(j0)} +\\
&\qquad + \beta_j \left(2\mu_i p_{3(ii)} + \sum_{u=1}^{i-1} \mu_u p_{3(ui)} + \sum_{u=i+1}^{r} \mu_u p_{3(iu)} \right) +\\
&\qquad + \beta_i \left(2\mu_j p_{3(jj)} + \sum_{u=1}^{j-1} \mu_u p_{3(uj)} + \sum_{u=j+1}^{r} \mu_u p_{3(ju)} \right) = 0,\\
&\qquad\qquad\qquad\qquad 1 \le i \le r-1, \ i+1 \le j \le r;\\
&-\left((m-k)\lambda + 2\mu_i\right) p_{k(ii)} + (m-k+1)\lambda p_{k-1(ii)} + 2\mu_i\beta_i p_{k+1(ii)} +\\
&\quad + \beta_i \left(\sum_{u=1}^{i-1} \mu_u p_{k+1(ui)} + \sum_{u=i+1}^{r} \mu_u p_{k+1(iu)} \right) = 0, \quad 3 \le k \le m-1, \quad 1 \le i \le r;\\
&-\left((m-k)\lambda + \mu_i + \mu_j\right) p_{k(ij)} + (m-k+1)\lambda p_{k-1(ij)} +\\
&\quad + \beta_j \left(2\mu_i p_{k+1(ii)} + \sum_{u=1}^{i-1} \mu_u p_{k+1(ui)} + \sum_{u=i+1}^{r} \mu_u p_{k+1(iu)} \right) +
\end{aligned}$$

$$+\beta_i\left(2\mu_j p_{k+1(jj)}+\sum_{u=1}^{j-1}\mu_u p_{k+1(uj)}+\sum_{u=j+1}^{r}\mu_u p_{k+1(ju)}\right)=0,$$
$$3\le k\le m-1,\ \ 1\le i\le r-1,\ \ i+1\le j\le r;$$
$$-2\mu_i p_{m(ii)}+\lambda p_{m-1(ii)}=0,\quad 1\le i\le r; \tag{3.1}$$
$$-(\mu_i+\mu_j)p_{m(ij)}+\lambda p_{m-1(ij)}=0,\quad 1\le i\le r-1,\ \ i+1\le j\le r;$$
$$p_0+\sum_{i=1}^{r}p_{1(i0)}+\sum_{k=2}^{m}\sum_{i=1}^{r}\sum_{j=i}^{r}p_{k(ij)}=1.$$

Solving the system (3.1), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$p_1=\sum_{i=1}^{r}p_{1(i0)};\quad p_k=\sum_{i=1}^{r}\sum_{j=i}^{r}p_{k(ij)},\quad 2\le k\le m.$$

4. Equations for steady-state probabilities of the three-channel closed system with H_r-distribution of service times

Let us enumerate the three-channel system's states as follows: x_0 corresponds to the empty system; $x_{k(ijs)}$ is the state, when there are k customers in the system ($1\le k\le m$), the service time is in the phase i ($1\le i\le r$), in the phase j ($i\le j\le r$) and in the phase s ($j\le s\le r$) for each of the three channels respectively. In the case when $k=1$, we assume that $j=s=0$, and for $k=2$, we take $s=0$. We denote by p_0 and $p_{k(ijs)}$ respectively, steady-state probabilities that the system is in the each of these states. To calculate p_0 and $p_{k(ijs)}$ we obtain the system of linear equations:

$$-m\lambda p_0+\sum_{i=1}^{r}\mu_i p_{1(i00)}=0;$$
$$-((m-1)\lambda+\mu_i)\,p_{1(i00)}+m\lambda\beta_i p_0+2\mu_i p_{2(ii0)}+\sum_{u=1}^{i-1}\mu_u p_{2(ui0)}+$$
$$+\sum_{j=i+1}^{r}\mu_j p_{2(ij0)}=0,\ \ 1\le i\le r;$$
$$-((m-2)\lambda+2\mu_i)\,p_{2(ii0)}+(m-1)\lambda\beta_i p_{1(i00)}+3\mu_i p_{3(iii)}+$$
$$+\sum_{s=1}^{i-1}\mu_s p_{3(iis)}+\sum_{s=i+1}^{r}\mu_s p_{3(iis)}=0,\quad 1\le i\le r;$$
$$-((m-2)\lambda+\mu_i+\mu_j)\,p_{2(ij0)}+(m-1)\lambda\beta_j p_{1(i00)}+(m-1)\lambda\beta_i p_{1(j00)}+$$
$$+2\mu_i p_{3(iij)}+\sum_{u=1}^{i-1}\mu_u p_{3(uij)}+\sum_{u=i+1}^{j-1}\mu_u p_{3(iuj)}+$$
$$+2\mu_j p_{3(jji)}+\sum_{u=j+1}^{r}\mu_u p_{3(iju)}=0,\ \ 1\le i\le r-1,\ \ i+1\le j\le r;$$

$$
\begin{aligned}
&-\left((m-3)\lambda+3\mu_i\right)p_{3(iii)}+(m-2)\lambda\beta_i p_{2(ii0)}+3\beta_i\mu_i p_{4(iii)}+\\
&\quad+\beta_i\left(\sum_{s=1}^{i-1}\mu_s p_{4(iis)}+\sum_{s=i+1}^{r}\mu_s p_{4(iis)}\right)=0,\quad 1\le i\le r;\\
&-\left((m-3)\lambda+2\mu_i+\mu_s\right)p_{3(iis)}+(m-2)\lambda\beta_i p_{2(is0)}+(m-2)\lambda\beta_s p_{2(ii0)}+\\
&\quad+\beta_s\left(3\mu_i p_{4(iii)}+\sum_{u=1}^{i-1}\mu_u p_{4(iiu)}+\sum_{u=i+1}^{r}\mu_u p_{4(iiu)}\right)+\beta_i\sum_{u=s+1}^{r}\mu_u p_{4(isu)}+\\
&\quad+\beta_i\left(2\mu_i p_{4(iis)}+2\mu_s p_{4(ssi)}+\sum_{u=1}^{i-1}\mu_u p_{4(uis)}+\sum_{u=i+1}^{s-1}\mu_u p_{4(ius)}\right)=0,\\
&\qquad\qquad\qquad 1\le i\le r-1,\ \ i+1\le s\le r;\\
&-\left((m-3)\lambda+2\mu_i+\mu_s\right)p_{3(iis)}+(m-2)\lambda\beta_i p_{2(si0)}+(m-2)\lambda\beta_s p_{2(ii0)}+\\
&\quad+\beta_s\left(3\mu_i p_{4(iii)}+\sum_{u=1}^{i-1}\mu_u p_{4(iiu)}+\sum_{u=i+1}^{r}\mu_u p_{4(iiu)}\right)+\beta_i\sum_{u=i+1}^{r}\mu_u p_{4(siu)}+\\
&\quad+\beta_i\left(2\mu_i p_{4(iis)}+2\mu_s p_{4(ssi)}+\sum_{u=1}^{s-1}\mu_u p_{4(usi)}+\sum_{u=s+1}^{i-1}\mu_u p_{4(sui)}\right)=0,\\
&\qquad\qquad\qquad 2\le i\le r,\ \ 1\le s\le i-1;\\
&-\left((m-3)\lambda+\mu_i+\mu_j+\mu_s\right)p_{3(ijs)}+(m-2)\lambda\beta_i p_{2(js0)}+(m-2)\lambda\beta_j p_{2(is0)}+\\
&+(m-2)\lambda\beta_s p_{2(ij0)}+\beta_i\left(\sum_{u=1}^{j-1}\mu_u p_{4(ujs)}+\sum_{u=j+1}^{s-1}\mu_u p_{4(jus)}+\sum_{u=s+1}^{r}\mu_u p_{4(jsu)}\right)+\\
&\quad+\beta_j\left(\sum_{u=1}^{i-1}\mu_u p_{4(uis)}+\sum_{u=i+1}^{s-1}\mu_u p_{4(ius)}+\sum_{u=s+1}^{r}\mu_u p_{4(isu)}\right)+\\
&\quad+\beta_s\left(\sum_{u=1}^{i-1}\mu_u p_{4(uij)}+\sum_{u=i+1}^{j-1}\mu_u p_{4(iuj)}+\sum_{u=j+1}^{r}\mu_u p_{4(iju)}\right)+\\
&\quad+2\beta_i\left(\mu_j p_{4(jjs)}+\mu_s p_{4(ssj)}\right)+2\beta_j\left(\mu_i p_{4(iis)}+\mu_s p_{4(ssi)}\right)+\\
&\quad+2\beta_s\left(\mu_i p_{4(iij)}+\mu_j p_{4(jji)}\right)=0,\\
&\qquad\qquad 1\le i\le r-2,\ \ i+1\le j\le r-1,\ \ j+1\le s\le r;\\
&-\left((m-k)\lambda+3\mu_i\right)p_{k(iii)}+(m-k+1)\lambda p_{k-1(iii)}+3\beta_i\mu_i p_{k+1(iii)}+\\
&\quad+\beta_i\left(\sum_{u=1}^{i-1}\mu_u p_{k+1(iiu)}+\sum_{u=i+1}^{r}\mu_u p_{k+1(iiu)}\right)=0,\ \ 4\le k\le m-1,\ \ 1\le i\le r;\\
&-\left((m-k)\lambda+2\mu_i+\mu_s\right)p_{k(iis)}+(m-k+1)\lambda p_{k-1(iis)}+\\
&\quad+3\mu_i\beta_s p_{k+1(iii)}+2\mu_i\beta_i p_{k+1(iis)}+2\mu_s\beta_i p_{k+1(ssi)}+\beta_s\sum_{u=1}^{i-1}\mu_u p_{k+1(iiu)}+\\
&\quad+\beta_s\sum_{u=i+1}^{r}\mu_u p_{k+1(iiu)}+\beta_i\sum_{u=1}^{i-1}\mu_u p_{k+1(uis)}+\\
&\quad+\beta_i\left(\sum_{u=i+1}^{s-1}\mu_u p_{k+1(ius)}+\sum_{u=s+1}^{r}\mu_u p_{k+1(isu)}\right)=0,\\
&\qquad 4\le k\le m-1,\ \ 1\le i\le r-1,\ \ i+1\le s\le r;
\end{aligned}
$$

$$
\begin{aligned}
&-((m-k)\lambda+2\mu_i+\mu_s))p_{k(iis)}+(m-k+1)\lambda p_{k-1(iis)}+\\
&+3\mu_i\beta_s p_{k+1(iii)}+2\mu_i\beta_i p_{k+1(iis)}+2\mu_s\beta_i p_{k+1(ssi)}+\beta_s\sum_{u=1}^{i-1}\mu_u p_{k+1(iiu)}+\\
&\quad+\beta_s\sum_{u=i+1}^{r}\mu_u p_{k+1(iiu)}+\beta_i\sum_{u=i+1}^{r}\mu_u p_{k+1(siu)}+\\
&\quad+\beta_i\left(\sum_{u=1}^{s-1}\mu_u p_{k+1(usi)}+\sum_{u=s+1}^{i-1}\mu_u p_{k+1(sui)}\right)=0,\\
&\qquad\qquad 4\le k\le m-1,\ \ 2\le i\le r,\ \ 1\le s\le i-1;\\
&-((m-k)\lambda+\mu_i+\mu_j+\mu_s)\,p_{k(ijs)}+(m-k+1)\lambda p_{k-1(ijs)}+\\
&\quad+\beta_i\left(\sum_{u=1}^{j-1}\mu_u p_{k+1(ujs)}+\sum_{u=j+1}^{s-1}\mu_u p_{k+1(jus)}+\sum_{u=s+1}^{r}\mu_u p_{k+1(jsu)}\right)+\\
&\quad+\beta_j\left(\sum_{u=1}^{i-1}\mu_u p_{k+1(uis)}+\sum_{u=i+1}^{s-1}\mu_u p_{k+1(ius)}+\sum_{u=s+1}^{r}\mu_u p_{k+1(isu)}\right)+\\
&\quad+\beta_s\left(\sum_{u=1}^{i-1}\mu_u p_{k+1(uij)}+\sum_{u=i+1}^{j-1}\mu_u p_{k+1(iuj)}+\sum_{u=j+1}^{r}\mu_u p_{k+1(iju)}\right)+\\
&\quad+2\beta_i\left(\mu_j p_{k+1(jjs)}+\mu_s p_{k+1(ssj)}\right)+2\beta_j\left(\mu_i p_{k+1(iis)}+\mu_s p_{k+1(ssi)}\right)+\\
&\quad+2\beta_s\left(\mu_i p_{k+1(iij)}+\mu_j p_{k+1(jji)}\right)=0,\\
&\quad 4\le k\le m-1,\ \ 1\le i\le r-2,\ \ i+1\le j\le r-1,\ \ j+1\le s\le r;\\
&-3\mu_i p_{m(iii)}+\lambda p_{m-1(iii)}=0,\quad 1\le i\le r;\\
&-(2\mu_i+\mu_s)p_{m(iis)}+\lambda p_{m-1(iis)}=0,\quad 1\le i\le r,\ \ 1\le s\le r,\ \ s\ne i;\\
&-(\mu_i+\mu_j+\mu_s)p_{m(ijs)}+\lambda p_{m-1(ijs)}=0,\\
&\qquad\qquad\quad 1\le i\le r-2,\ \ i+1\le j\le r-1,\ \ j+1\le s\le r;\\
&p_0+\sum_{i=1}^{r}p_{1(i00)}+\sum_{i=1}^{r}\sum_{j=i}^{r}p_{2(ij0)}+\sum_{k=3}^{m}\sum_{i=1}^{r}\sum_{s=1}^{r}p_{k(iis)}+\\
&\qquad\qquad\qquad+\sum_{k=3}^{m}\sum_{i=1}^{r-2}\sum_{j=i+1}^{r-1}\sum_{s=j+1}^{r}p_{k(ijs)}=1.
\end{aligned}
\tag{4.1}
$$

Solving the system (4.1), we find the steady-state probabilities p_k of the presence in the queueing system of k customers using the formulas

$$
\begin{aligned}
&p_1=\sum_{i=1}^{r}p_{1(i00)},\quad p_2=\sum_{i=1}^{r}\sum_{j=i}^{r}p_{2(ij0)};\\
&p_k=\sum_{i=1}^{r}\sum_{s=1}^{r}p_{k(iis)}+\sum_{i=1}^{r-2}\sum_{j=i+1}^{r-1}\sum_{s=j+1}^{r}p_{k(ijs)},\quad 3\le k\le m.
\end{aligned}
$$

5. Features of finding probabilities p_k in the case of complex-valued or paradoxical parameters of H_r-approximation

We calculate the approximate steady-state probabilities p_k for n-channel closed systems with $n = 1$, $n = 2$ and $n = 3$, and arbitrary distribution of service times using solutions of equations (2.1), (3.1) and (4.1) respectively, written for the closed systems with H_r-distribution of service times, considering the order of approximation r from 2 to 6.

The system of equations of the moments method for approximating the distribution of some random variable X using a random variable Y_r, distributed by law of H_r, is of the form

$$\sum_{s=1}^{r} \frac{\beta_s}{\mu_s^i} = \frac{m_i}{i!}, \qquad 0 \le i \le 2r-1; \qquad \sum_{s=1}^{r} \beta_s = 1, \tag{5.1}$$

where $m_i = E(X^i)$ is the initial moment of order i of the random variable X. The dependence of the nature of the roots of system (5.1) on values of the variation coefficient V for the original gamma distributions and Weibull distributions is described in [6]. For the values $V < 1$ of the variation coefficient, some of the roots of system (5.1) are complex-valued but in most cases as a result of summation of probabilities of microstates the steady-state probabilities p_k are real-valued.

To illustrate this fact, we consider the solutions of system (2.1) for complex-valued parameters β_s and μ_s, limited to the case when $r = 2$ and $m = 2$. In this case, using the solutions of system (2.1) and formula (2.2), we obtain

$$\begin{aligned} p_0 &= \frac{\mu_1\mu_2}{\Delta}\big(\lambda(\beta_1\mu_1 + \beta_2\mu_2) + \mu_1\mu_2\big), \\ p_1 &= \frac{2\lambda\mu_1\mu_2}{\Delta}(\lambda + \beta_2\mu_1 + \beta_1\mu_2), \quad p_2 = 1 - p_0 - p_1, \\ \Delta &= -2\lambda^3(\beta_2\mu_1 + \beta_1\mu_2) + 2\lambda^2(\beta_2\mu_1^2 + \beta_1\mu_2^2 + \mu_1\mu_2) + \\ &\quad + \lambda\mu_1\mu_2\big((\beta_1 + 2\beta_2)\mu_1 + (\beta_2 + 2\beta_1)\mu_2\big) + \mu_1^2\mu_2^2. \end{aligned} \tag{5.2}$$

If parameters β_s and μ_s ($s = 1, 2$) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to such two:

$$\begin{aligned} &1)\ \beta_1 = a + ib,\ \ \mu_1 = c + id; \quad \beta_2 = a - ib,\ \ \mu_2 = c - id; \\ &2)\ \beta_1 = a + ib,\ \ \mu_1 = c - id; \quad \beta_2 = a - ib,\ \ \mu_2 = c + id. \end{aligned} \tag{5.3}$$

In each of these cases, the imaginary parts in expressions (5.2) for p_k ($k = 0, 1, 2$) are reduced, because the expressions

$$\mu_1\mu_2, \quad \beta_1\mu_2 + \beta_2\mu_1, \quad \beta_1\mu_1 + \beta_2\mu_2, \quad \beta_1\mu_2^2 + \beta_2\mu_1^2$$

of which consist p_k, are real-valued.

In the case of complex-valued or paradoxical roots β_s and μ_s of system (5.1), let us name the function $F_{H_r}(t) = 1 - \sum\limits_{s=1}^{r} \beta_s e^{-\mu_s t}$ $(t \ge 0)$ the distribution pseudo-function by law of H_r. Let us show that the function $F_{H_r}(t)$ is a real-valued function if β_s and μ_s $(1 \le s \le r)$ are roots of system (5.1).

In fact, if some of the roots of system (5.1) are complex-valued, then they can only be complex conjugate, and all possible cases of alternation of signs before the imaginary unit can be reduced to two cases presented in (5.3). In each of these cases, the imaginary parts in the expression for $F_{H_r}(t)$ are reduced, so the result is the real-valued function:

$$\begin{aligned}
&1)\;\; \beta_1 e^{-\mu_1 t} + \beta_2 e^{-\mu_2 t} = 2\, e^{-ct}\left(a \cdot \cos(dt) + b \cdot \sin(dt)\right);\\
&2)\;\; \beta_1 e^{-\mu_1 t} + \beta_2 e^{-\mu_2 t} = 2\, e^{-ct}\left(a \cdot \cos(dt) - b \cdot \sin(dt)\right).
\end{aligned}$$

The absolute deviation of the function of distribution by law G from a function $F_{H_r}(t)$ which parameters are roots of system (5.1), we will evaluate with the help of integral

$$\Delta_l(F) = \int_0^{\infty} |F_{H_r}(t) - F_G(t)| dt,$$

where $F_G(t)$ is the probability distribution function by law G.

Let $\Gamma(V)$, $W(V)$ and $U[a,\, b]$ denote the gamma distribution, Weibull distribution with coefficients of variation V, and uniform distribution on the interval $[a,\, b]$, respectively.

In Table 1, we give values of deviation $\Delta_r(F)$ for $r = 2, \dots, 6$, calculated by results of approximation of different distributions with means 0.5. With increasing order of H_r-distribution, the value of deviation $\Delta_r(F)$ decreases, and with the increase of the variation coefficient for $V > 1$, the deviation increases, much faster for the Weibull distribution compared with the gamma distribution. The deviations $\Delta_r(F)$ have the smallest values for the distribution $\Gamma(0.7)$. For the distributions $W(0.7)$, $W(0.8)$, $W(0.9)$ and $W(0.95)$ for some values of r, the deviation $\Delta_r(F) = \infty$. In each of these cases, one of roots μ_s of system (5.1) is real, but negative. Therefore, for the corresponding distribution pseudofunction, the limit relation $\lim\limits_{t\to\infty} F_{H_r}(t) = \infty$ is valid. For these values of r, the steady-state probabilities p_k, obtained using solutions of equations (2.1), (3.1) and (4.1), written for the closed systems with H_r-distribution of service times, can be paradoxical or more differ from the exact values.

For the distribution $U[0.49913,\, 0.50087]$, the coefficient of variation $V = 0.001$. The distributions $U[0.49913,\, 0.50087]$ and $\Gamma(0.001)$ are close to the degenerate distribution with $V = 0$, therefore, the initial moments, roots of equations (5.1) and deviations $\Delta_r(F)$ of these distributions practically coincide. For convenience, we introduce a new designation $U(0.001)$ for distribution $U[0.49913,\, 0.50087]$.

In Table 2, we present information about properties of the roots of system (5.1) for different distributions in the case when $r = 6$. For all represented distributions real parts of complex roots μ_s are positive. For distributions with coefficient of variation $V > 1$, all roots of system (5.1) are real, positive and non-paradoxical. Calculations show that the properties of solutions of systems (2.1), (3.1) and (4.1) in the sense of their signs and that there are real or complex ones among them, basically repeat the properties of the roots β_s $(1 \le s \le r)$ of system (5.1).

Table 1. Values of the absolute deviation $\Delta_r(F)$ for different distributions

Distribution name	$\Delta_2(F)$	$\Delta_3(F)$	$\Delta_4(F)$	$\Delta_5(F)$	$\Delta_6(F)$
$\Gamma(0.001)$	0.1811	0.1304	0.1046	0.0888	0.0773
$U[0.49913,\ 0.50087]$	0.1810	0.1302	0.1046	0.0886	0.0773
$U[0,\ 1]$	0.0570	0.0316	0.0205	0.0147	0.0112
$\Gamma(0.3)$	0.0718	0.0240	$7.2 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$	$4.0 \cdot 10^{-4}$
$W(0.3)$	0.0770	0.0310	0.0135	$6.2 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$
$\Gamma(0.7)$	$3.7 \cdot 10^{-4}$	$3.6 \cdot 10^{-5}$	$6.8 \cdot 10^{-6}$	$1.9 \cdot 10^{-6}$	$6.4 \cdot 10^{-7}$
$W(0.7)$	$3.6 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$2.8 \cdot 10^{-4}$	∞	$3.1 \cdot 10^{-5}$
$W(0.8)$	$2.1 \cdot 10^{-3}$	∞	$2.2 \cdot 10^{-4}$	$6.2 \cdot 10^{-5}$	∞
$W(0.9)$	$2.5 \cdot 10^{-3}$	$2.6 \cdot 10^{-4}$	∞	$5.5 \cdot 10^{-5}$	$2.4 \cdot 10^{-5}$
$W(0.95)$	$1.6 \cdot 10^{-3}$	$2.7 \cdot 10^{-4}$	$6.5 \cdot 10^{-5}$	∞	$1.8 \cdot 10^{-5}$
$\Gamma(0.9)$	$2.7 \cdot 10^{-3}$	$5.6 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	$6.8 \cdot 10^{-5}$	$3.1 \cdot 10^{-5}$
$\Gamma(1.5)$	0.0307	0.0109	$5.1 \cdot 10^{-3}$	$2.7 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
$\Gamma(2)$	0.0658	0.0265	0.0135	$7.9 \cdot 10^{-3}$	$5.1 \cdot 10^{-3}$
$\Gamma(4)$	0.1573	0.0706	0.0393	0.0248	0.0170
$W(3)$	0.1986	0.1395	0.1085	0.0893	0.0762

Table 2. Properties of the roots of system (5.1) for different distributions (case when $r = 6$)

Distribution name	Number of complex roots β_s	Number of complex roots μ_s	Number of β_s with $\mathrm{Re}\,\beta_s < 0$	Number of real $\beta_s \notin [0,\ 1]$	Number of real $\mu_s < 0$
$\Gamma(0.001)$	6	6	4	0	0
$U(0.001)$	6	6	4	0	0
$U[0,\ 1]$	6	6	2	0	0
$\Gamma(0.3)$	6	6	4	0	0
$W(0.3)$	6	6	2	0	0
$\Gamma(0.7)$	2	2	0	0	0
$W(0.7)$	4	4	2	2	0
$W(0.8)$	4	4	2	2	1
$W(0.9)$	2	2	2	4	0
$W(0.95)$	2	2	0	3	0
$\Gamma(0.9)$	0	0	0	6	0

6. Numerical results

To identify closed systems, we introduce the standard notation used for queueing systems. Let us present the results of calculating steady-state probabilities on examples of the $M/U(0.001)/n/10$, $M/U[0,\ 1]/n/10$, $M/\Gamma(0.7)/n/10$, $M/W(0.9)/n/10$, $M/\Gamma(1.5)/n/10$, $M/\Gamma(2)/n/10$ and $M/\Gamma(4)/n/10$ closed systems in the cases when $n = 1$, $n = 2$ and $n = 3$. We assume that customers arrive to the queueing system from $m = 10$ identical sources.

We take $\lambda = 1$ and $E(T_{\mathrm{sv}}) = 0.5$, where $E(T_{\mathrm{sv}})$ denote the mean of the service times.

For single-channel systems, we test the obtained results using the method of potentials [8], which allows us to calculate the steady-state probabilities p_k of the presence k customers in the $M/G/1/m$ closed queueing system.

For cases of two and three channels, the obtained results are verified using simulation models constructed with the help of the GPSS World tools [7]. The results obtained using GPSS World slightly differ from one another for different numbers of library random-number generators used for simulating random variables. Therefore, we use averaged results obtained using simulation models with different values of random-numbers generators that take on values of natural numbers from 6 to 10.

Let us introduce the designation: N and σ are the mean and standard deviation of the number of customers in a closed queueing system respectively, and

$$\Delta_{(r,r-1)} = \sum_{k=0}^{10} |p_{k(r)} - p_{k(r-1)}|, \qquad \Delta_{(6,r)} = \sum_{k=0}^{10} |p_{k(6)} - p_{k(r)}|,$$
$$\Delta_{r(Pot)} = \sum_{k=0}^{10} |p_{k(r)} - p_{k(Pot)}|, \qquad \Delta_{r(sim)} = \sum_{k=0}^{10} |p_{k(r)} - p_{k(sim)}|,$$
$$p_{k(sim)} = \frac{1}{5}\sum_{i=6}^{10} p_{k(sim,i)}, \qquad 0 \le k \le 10, \quad 2 \le r \le 6.$$

Here $p_{k(Pot)}$ and $p_{k(r)}$ are values of probabilities p_k, obtained using the method of potentials and H_r-approximation respectively ($p_{k(Pot)} = p_k$); $p_{k(sim)}$ is the average value of probabilities $p_{k(sim,i)}$, obtained by means of the simulation model using the number i of random-numbers generator for $i \in \{6, \dots, 10\}$. Thus, the quantities $\Delta_{r(Pot)}$ and $\Delta_{r(sim)}$ are measures of deviations of the distributions $\{p_{k(r)}\}$ from distributions $\{p_{k(Pot)}\}$ and $\{p_{k(sim)}\}$, respectively, and the quantities $\Delta_{(r,r-1)}$ and $\Delta_{(6,r)}$ give an opportunity to estimate the deviation of distributions $\{p_{k(r)}\}$ from distributions $\{p_{k(r-1)}\}$ and $\{p_{k(6)}\}$, respectively. Simulation time for GPSS World is taken equal to $5 \cdot 10^6$.

In Table 3 and 4 we have results of the calculation of steady-state characteristics of the $M/G/1/10$ and $M/G/n/10$ ($n = 2, 3$) closed systems respectively, with considered gamma, Weibull and uniform distributions of service times.

The values of $\Delta_{r(Pot)}$ and $\Delta_{(6,r)}$ ($2 \le r \le 5$) in Table 3 are either identical or at least are numbers of the same order. Note that only for the distribution $\Gamma(4)$ the deviations $\Delta_{5(Pot)}$ and $\Delta_{(6,5)}$ are numbers that differ in one order. This means that in most cases we can use values $\Delta_{(6,r)}$ to evaluate the accuracy of the approximation of the distribution $\{p_{k(r)}\}$ to the true $\{p_k\}$ for $r \in \{2, \dots, 5\}$.

The results presented in Table 3, indicate that the values of absolute deviations $\Delta_{r(Pot)}$ and $\Delta_{(6,r)}$ decrease with increasing order of H_r-distributions in approximations, as well as the values of $\Delta_{(r,r-1)}$, which decrease with increasing of r, means that the values of distribution $\{p_{k(r)}\}$ with each step getting closer to a true distribution $\{p_k\}$. With the growth of the variation coefficient of distributions after the value of $V > 1$, as expected taking into account the behavior of deviations $\Delta_r(F)$, the values of the absolute deviations $\Delta_{r(Pot)}$ and $\Delta_{(6,r)}$ also increase. For the distribution $W(0.9)$ the deviation $\Delta_4(F) = \infty$ and,

Table 3. Results of the calculation of steady-state characteristics of the $M/G/1/10$ closed systems with different G-distributions

G-distribuion name	Characteristic name	Method of calculation and values of characteristics					
		H_r-approximation					Potentials
		$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	
$U(0.001)$	N	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
	$\Delta_{r(Pot)}$	0.0111	0.0003	$4 \cdot 10^{-6}$	$7 \cdot 10^{-8}$	$2 \cdot 10^{-9}$	–
	$\Delta_{(r,r-1)}$	–	0.0111	0.0003	$4 \cdot 10^{-6}$	$7 \cdot 10^{-8}$	–
	$\Delta_{(6,r)}$	0.0111	0.0003	$4 \cdot 10^{-6}$	$7 \cdot 10^{-8}$	–	–
$U[0,\ 1]$	N	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
	$\Delta_{r(Pot)}$	0.0084	0.0004	$2 \cdot 10^{-5}$	$6 \cdot 10^{-7}$	$2 \cdot 10^{-8}$	–
	$\Delta_{(r,r-1)}$	–	0.0084	0.0004	$2 \cdot 10^{-5}$	$6 \cdot 10^{-7}$	–
	$\Delta_{(6,r)}$	0.0084	0.0004	$2 \cdot 10^{-5}$	$6 \cdot 10^{-7}$	–	–
$\Gamma(0.7)$	N	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
	$\Delta_{r(Pot)}$	$7 \cdot 10^{-5}$	$8 \cdot 10^{-7}$	$2 \cdot 10^{-8}$	$4 \cdot 10^{-10}$	$1 \cdot 10^{-11}$	–
	$\Delta_{(r,r-1)}$	–	$7 \cdot 10^{-5}$	$8 \cdot 10^{-7}$	$2 \cdot 10^{-8}$	$4 \cdot 10^{-10}$	–
	$\Delta_{(6,r)}$	$7 \cdot 10^{-5}$	$8 \cdot 10^{-7}$	$2 \cdot 10^{-8}$	$4 \cdot 10^{-10}$	–	–
$W(0.9)$	N	8.0000	8.0000	8.0000	8.0000	8.0000	8.0000
	$\Delta_{r(Pot)}$	0.0007	$6 \cdot 10^{-6}$	$1 \cdot 10^{-5}$	$6 \cdot 10^{-8}$	$4 \cdot 10^{-9}$	–
	$\Delta_{(r,r-1)}$	–	0.0007	$1 \cdot 10^{-5}$	$1 \cdot 10^{-5}$	$5 \cdot 10^{-8}$	–
	$\Delta_{(6,r)}$	0.0007	$6 \cdot 10^{-6}$	$1 \cdot 10^{-5}$	$5 \cdot 10^{-8}$	–	–
$\Gamma(1.5)$	N	8.0012	8.0017	8.0018	8.0018	8.0018	8.0018
	$\Delta_{r(Pot)}$	0.0130	0.0012	0.0002	$2 \cdot 10^{-5}$	$7 \cdot 10^{-6}$	–
	$\Delta_{(r,r-1)}$	–	0.0127	0.0011	0.0001	$2 \cdot 10^{-5}$	–
	$\Delta_{(6,r)}$	0.0130	0.0012	0.0002	$2 \cdot 10^{-5}$	–	–
$\Gamma(2)$	N	8.0046	8.0084	8.0094	8.0096	8.0097	8.0097
	$\Delta_{r(Pot)}$	0.0356	0.0053	0.0011	0.0004	0.0001	–
	$\Delta_{(r,r-1)}$	–	0.0337	0.0046	0.0010	0.0003	–
	$\Delta_{(6,r)}$	0.0357	0.0053	0.0011	0.0003	–	–
$\Gamma(4)$	N	8.0285	8.0701	8.0847	8.0894	8.0910	8.0921
	$\Delta_{r(Pot)}$	0.1381	0.0587	0.0234	0.0112	0.0062	–
	$\Delta_{(r,r-1)}$	–	0.1237	0.0397	0.0148	0.0063	–
	$\Delta_{(6,r)}$	0.1383	0.0577	0.0201	0.0063	–	–

consequently, we have that $\Delta_{(6,4)} > \Delta_{(6,3)}$, that is, for $r = 4$ convergence to the true distribution slows down.

We do not represent in Table 3 values of deviations $\Delta_{r(sim)}$, but we can note that these values are no less than 10^{-4} for any distributions. Only for the $\Gamma(4)$ distribution the deviation of the distribution $\{p_{k(sim)}\}$ from the true distribution $\{p_k\}$ is smaller than this deviation for the distribution $\{p_{k(r)}\}$. In cases where the deviation $\Delta_{(6,5)}$ is less than 10^{-2}, the deviation of the distribution $\{p_{k(r)}\}$ from the true distribution $\{p_k\}$ and deviation $\Delta_{(r+1,r)}$ are numbers of the same order. Thus, in these cases we can use values $\Delta_{(r,r-1)}$ to evaluate accuracy of the approximation of the distribution $\{p_{k(r-1)}\}$ to the true $\{p_k\}$ for $3 \le r \le 6$. In cases where $\Delta_{(r,r-1)} < 10^{-4}$, we can argue that the distribution $\{p_{k(r-1)}\}$ is more accurate approximation than $\{p_{k(sim)}\}$.

Table 4. Results of the calculation of steady-state characteristics of the $M/G/n/10$ closed systems with different G-distributions for $n = 2$ and $n = 3$

G-distribuion name, n	Characteristic name	Method of calculation and values of characteristics					
		H_r-approximation					GPSS World
		$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	
$U(0.001)$, $n = 2$	N	6.0027	6.0036	6.0035	6.0035	6.0035	6.0022
	$\Delta_{r(sim)}$	0.0128	0.0091	0.0092	0.0090	0.0071	–
	$\Delta_{(r,r-1)}$	–	0.0071	0.0003	0.0002	0.0019	–
$U(0.001)$, $n = 3$	N	4.3266	4.3309	4.3283	4.3269	4.3257	4.3253
	$\Delta_{r(sim)}$	0.0175	0.0125	0.0072	0.0040	0.0015	–
	$\Delta_{(r,r-1)}$	–	0.0239	0.0096	0.0046	0.0026	–
$U[0, 1]$, $n = 2$	N	6.0136	6.0149	6.0148	6.0148	6.0148	6.0144
	$\Delta_{r(sim)}$	0.0056	0.0008	0.0005	0.0005	0.0005	–
	$\Delta_{(r,r-1)}$	–	0.0059	0.0005	0.0001	$3 \cdot 10^{-5}$	–
$U[0, 1]$, $n = 3$	N	4.4038	4.4076	4.4071	4.4073	4.4072	4.4071
	$\Delta_{r(sim)}$	0.0094	0.0022	0.0011	0.0006	0.0005	–
	$\Delta_{(r,r-1)}$	–	0.0110	0.0031	0.0012	0.0004	–
$\Gamma(0.7)$, $n = 2$	N	6.0171	6.0171	6.0172	6.0172	6.0172	6.0164
	$\Delta_{r(sim)}$	0.0008	0.0008	0.0008	0.0008	0.0008	–
	$\Delta_{(r,r-1)}$	–	$5 \cdot 10^{-5}$	$1 \cdot 10^{-6}$	$9 \cdot 10^{-8}$	$1 \cdot 10^{-8}$	–
$\Gamma(0.7)$, $n = 3$	N	4.4230	4.4230	4.4230	4.4230	4.4230	4.4224
	$\Delta_{r(sim)}$	0.0006	0.0005	0.0005	0.0005	0.0005	–
	$\Delta_{(r,r-1)}$	–	0.0001	$8 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-7}$	–
$W(0.9)$, $n = 2$	N	6.0302	6.0301	6.0301	6.0301	6.0301	6.0297
	$\Delta_{r(sim)}$	0.0008	0.0004	0.0004	0.0004	0.0004	–
	$\Delta_{(r,r-1)}$	–	0.0005	$6 \cdot 10^{-6}$	$2 \cdot 10^{-5}$	$1 \cdot 10^{-6}$	–
$W(0.9)$, $n = 3$	N	4.4714	4.4710	4.4710	4.4710	4.4710	4.4706
	$\Delta_{r(sim)}$	0.0009	0.0005	0.0005	0.0005	0.0005	–
	$\Delta_{(r,r-1)}$	–	0.0010	$1 \cdot 10^{-5}$	$9 \cdot 10^{-5}$	$7 \cdot 10^{-6}$	–
$\Gamma(1.5)$, $n = 2$	N	6.0848	6.0898	6.0905	6.0907	6.0907	6.0906
	$\Delta_{r(sim)}$	0.0145	0.0030	0.0012	0.0009	0.0008	–
	$\Delta_{(r,r-1)}$	–	0.0124	0.0021	0.0006	0.0002	–
$\Gamma(1.5)$, $n = 3$	N	4.6036	4.6119	4.6131	4.6134	4.6135	4.6139
	$\Delta_{r(sim)}$	0.0216	0.0066	0.0025	0.0010	0.0005	–
	$\Delta_{(r,r-1)}$	–	0.0174	0.0045	0.0016	0.0006	–
$\Gamma(2)$, $n = 2$	N	6.1347	6.1519	6.1552	6.1560	6.1562	6.1570
	$\Delta_{r(sim)}$	0.0473	0.0126	0.0043	0.0022	0.0012	–
	$\Delta_{(r,r-1)}$	–	0.0399	0.0095	0.0030	0.0012	–
$\Gamma(2)$, $n = 3$	N	4.6914	4.7160	4.7207	4.7220	4.7224	4.7233
	$\Delta_{r(sim)}$	0.0624	0.0212	0.0092	0.0042	0.0022	–
	$\Delta_{(r,r-1)}$	–	0.0478	0.0146	0.0055	0.0024	–
$\Gamma(4)$, $n = 2$	N	6.2591	6.3473	6.3742	6.3829	6.3861	6.3878
	$\Delta_{r(sim)}$	0.2295	0.0967	0.0446	0.0239	0.0136	–
	$\Delta_{(r,r-1)}$	–	0.1696	0.0604	0.0248	0.0115	–
$\Gamma(4)$, $n = 3$	N	4.8694	4.9744	5.0056	5.0158	5.0196	5.0212
	$\Delta_{r(sim)}$	0.2486	0.1090	0.0540	0.0305	0.0174	–
	$\Delta_{(r,r-1)}$	–	0.1768	0.0641	0.0276	0.0135	–

Table 5. Values of $\Delta_6(F)$, σ and $\Delta_{(6,5)}$ for the $M/W(V)/n/10$ closed systems with different V, for $\lambda = 1$, $n = 1$, 2, 3, $\lambda = 2$, $n = 2$ and $\lambda = 3$, $n = 3$

V	$\Delta_6(F)$	Characteristic name	Values of σ and $\Delta_{(6,5)}$ for different n and λ				
			$n=1$, $\lambda=1$	$n=2$, $\lambda=1$	$n=3$, $\lambda=1$	$n=2$, $\lambda=2$	$n=3$, $\lambda=3$
0.1	0.0364	σ	1.05	1.47	1.60	1.08	1.12
		$\Delta_{(6,5)}$	$7 \cdot 10^{-8}$	$2 \cdot 10^{-5}$	$\mathbf{6 \cdot 10^{-4}}$	$4 \cdot 10^{-7}$	$9 \cdot 10^{-7}$
0.2	0.0115	σ	1.06	1.49	1.62	1.10	1.13
		$\Delta_{(6,5)}$	$6 \cdot 10^{-8}$	$7 \cdot 10^{-6}$	$\mathbf{3 \cdot 10^{-4}}$	$3 \cdot 10^{-7}$	$1 \cdot 10^{-6}$
0.3	0.0029	σ	1.08	1.52	1.65	1.12	1.15
		$\Delta_{(6,5)}$	$6 \cdot 10^{-8}$	$4 \cdot 10^{-6}$	$\mathbf{2 \cdot 10^{-4}}$	$3 \cdot 10^{-7}$	$9 \cdot 10^{-7}$
0.4	$6 \cdot 10^{-4}$	σ	1.11	1.57	1.68	1.15	1.18
		$\Delta_{(6,5)}$	$5 \cdot 10^{-8}$	$2 \cdot 10^{-6}$	$6 \cdot 10^{-5}$	$2 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
0.5	$9 \cdot 10^{-5}$	σ	1.15	1.62	1.73	1.18	1.21
		$\Delta_{(6,5)}$	$4 \cdot 10^{-8}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-5}$	$1 \cdot 10^{-7}$	$4 \cdot 10^{-7}$
0.6	$3 \cdot 10^{-5}$	σ	1.20	1.68	1.77	1.23	1.25
		$\Delta_{(6,5)}$	$4 \cdot 10^{-8}$	$6 \cdot 10^{-7}$	$8 \cdot 10^{-6}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
0.7	$3 \cdot 10^{-5}$	σ	1.25	1.74	1.83	1.27	1.29
		$\Delta_{(6,5)}$	$6 \cdot 10^{-8}$	$2 \cdot 10^{-6}$	$3 \cdot 10^{-5}$	$3 \cdot 10^{-7}$	$1 \cdot 10^{-6}$
0.8	∞	σ	1.30	1.81	1.88	1.32	1.33
		$\Delta_{(6,5)}$	$4 \cdot 10^{-8}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-5}$	$2 \cdot 10^{-7}$	$9 \cdot 10^{-6}$
0.9	$2 \cdot 10^{-5}$	σ	1.36	1.87	1.93	1.36	1.37
		$\Delta_{(6,5)}$	$5 \cdot 10^{-8}$	$1 \cdot 10^{-6}$	$7 \cdot 10^{-6}$	$2 \cdot 10^{-7}$	$6 \cdot 10^{-7}$
1.1	$1 \cdot 10^{-4}$	σ	1.47	2.01	2.04	1.46	1.45
		$\Delta_{(6,5)}$	$1 \cdot 10^{-6}$	$8 \cdot 10^{-6}$	$4 \cdot 10^{-5}$	$3 \cdot 10^{-6}$	$7 \cdot 10^{-6}$
1.2	$4 \cdot 10^{-4}$	σ	1.53	2.07	2.09	1.51	1.50
		$\Delta_{(6,5)}$	$9 \cdot 10^{-6}$	$3 \cdot 10^{-5}$	$\mathbf{1 \cdot 10^{-4}}$	$2 \cdot 10^{-5}$	$4 \cdot 10^{-5}$
1.3	0.0010	σ	1.58	2.13	2.14	1.56	1.54
		$\Delta_{(6,5)}$	$3 \cdot 10^{-5}$	$\mathbf{1 \cdot 10^{-4}}$	$\mathbf{3 \cdot 10^{-4}}$	$7 \cdot 10^{-5}$	$\mathbf{1 \cdot 10^{-4}}$
1.4	0.0019	σ	1.64	2.22	2.21	1.64	1.61
		$\Delta_{(6,5)}$	$9 \cdot 10^{-5}$	$\mathbf{2 \cdot 10^{-4}}$	$\mathbf{6 \cdot 10^{-4}}$	$\mathbf{2 \cdot 10^{-4}}$	$\mathbf{3 \cdot 10^{-4}}$

If we follow criterion $\Delta_{(6,5)} < 10^{-4}$, for determining the accuracy of the distributions $\{p_{k(r)}\}$ obtained by the method of H_r-approximation, then according to the results presented in Table 4 for systems with two and three channels, we conclude that the distribution $\{p_{k(6)}\}$ is more accurate approximation than $\{p_{k(sim)}\}$ for the $M/U[0,\,1]/2/10$, $M/\Gamma(0.7)/2/10$, $M/\Gamma(0.7)/3/10$, $M/W(0.9)/2/10$ and $M/W(0.9)/3/10$ systems. For the rest of the considered systems, except for systems with the $\Gamma(4)$ distributions, the accuracy of the results obtained by both methods is comparable. For single-channel systems (see Table 3), inequality $\Delta_{(6,5)} > 10^{-4}$ is satisfied only for the cases of $\Gamma(2)$ and $\Gamma(4)$ distributions. Thus, an increase in the number of channels leads to a decrease in the accuracy of the obtained approximate distribution $\{p_{k(r)}\}$.

However, we note that the results, presented in Tables 3 and 4, were obtained for constant values of $\lambda = 1$ and $E(T_{\rm sv}) = 0.5$, therefore the system load factor $\rho = \lambda E(T_{\rm sv})/n$ varies with the number of channels n: $\rho = 0.5$, 0.25 and 1/6 for

$n = 1$, 2 and 3, respectively. Let us study the influence of the load factor on the accuracy of the obtained results using the example of systems with service times distributed according to the Weibull law. In Table 5 we have values of $\Delta_6(F)$, σ and $\Delta_{(6,5)}$ for the $M/W(V)/n/10$ closed systems with different coefficients of variation V, for $\lambda = 1$, $n = 1$, 2, 3, $\lambda = 2$, $n = 2$ and $\lambda = 3$, $n = 3$. In cases when $\lambda = n = 1$, $\lambda = n = 2$ and $\lambda = n = 3$ the load factor remains unchanged: $\rho = 0.5$.

We see that for a fixed value of the variation coefficient V, an increase in ρ leads to an increase in the standard deviation σ and deviation $\Delta_{(6,5)}$. At the same time, for a fixed value of V and ρ, an increase in the number of channels n leads to an insignificant increase in σ and $\Delta_{(6,5)}$.

The data in Table 5 allows us to analyze the influence of changes in the variation coefficient V on values of $\Delta_6(F)$, σ and $\Delta_{(6,5)}$. The minimum values of the deviations $\Delta_6(F)$ and $\Delta_{(6,5)}$ are achieved when $V = 0.6$. With increasing V for values $V > 1$ the values of $\Delta_6(F)$, σ and $\Delta_{(6,5)}$ increase sharply. For $V = 0.8$ we have that $\Delta_6(F) = \infty$, but only in the case $\lambda = n = 3$ this leads to a slight increase in the value of $\Delta_{(6,5)}$.

The values of $\Delta_{(6,5)}$ in Table 5 for which the inequality $\Delta_{(6,5)} > 10^{-4}$ holds, are shown in bold. For these cases the accuracy of the results obtained by the method of H_r-approximation and using GPSS World is comparable. For most cases, the distribution $\{p_{k(6)}\}$ is more accurate approximation than $\{p_{k(sim)}\}$. Calculations show that for most cases, the results obtained for the $M/\Gamma(V)/n/10$ closed systems with gamma distributions of service times are more accurate than for Weibull distributions.

7. Conclusions

This paper shows that the application of hyperexponential approximation of the service times distributions allows us to calculate steady-state probability distributions of the $M/G/n/m$ closed queueing systems for $n = 1$, 2 and 3 with high accuracy. For G-distributions with variation coefficient satisfying the condition $V < 1.4$, the accuracy can by higher than in the case of using simulation models. The accuracy of finding the steady-state probability distributions depends on both the variation coefficient V of service times and the system load factor ρ. We find these probability distributions using solutions of a system of linear algebraic equations obtained by the method of fictitious phases.

To obtain parameters of H_r-approximation of a certain distribution it is necessary to solve the system of equations of the moments method. For the values $V < 1$ of the variation coefficient, some of the roots of this system are complex-valued or, having a sense of probabilities, go beyond the interval [0, 1], but in most cases the final result is close to the desired distribution $\{p_k\}$.

Computing deviations $\Delta_{(r,r-1)}$ allows us to track the accuracy of approaching distributions $\{p_{k(r-1)}\}$ to the true distribution $\{p_k\}$ without the need to use simulation models.

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