Transactions of Azerbaijan National Academy of Sciences, Series of Physical-Technical and Mathematical Sciences: Informatics and Control Problems. – 2009. – Vol. XXIX, № 6. – P. 50-58.

S. A. ALIEV, Y. I. YELEYKO, Y. V. ZHERNOVYI

STEADY-STATE DISTRIBUTIONS FOR CERTAIN MODIFICATIONS OF THE M/M/1/m QUEUEING SYSTEM

For the M/M/1/m queueing system with blocking of an input flow from the moment of the beginning of the second service in a row to the moment of free system such cases are examined: 1) breaks in work are impossible (problem I); 2) after n services in a row a break in work of system occurs (problem II). For the problem I and for the problem II as $m \in [n, 2n]; m = n-1; n = 1$ *ergodic distributions of probabilities of states of system and of length of queue are obtained, and for* $m \geq n$ *an algorithm for finding of ergodic distribution is constructed. In the case of the M/M/1/*∞ *system conditions of existence of ergodic process for the problem I and for the problem II as* $n = 1$ *are defined. An another formulation of the problem I when blocking of an input flow continues from the start of the third service in a row is examined.*

1. Introduction. The queueing systems of type M/G/1/m with limited queue and regenerating level of input flow are examined in works [1, 2]. If the length of queue reaches number *m* then arrival of customers in system is blocked and renews only when the length of queue decreases up to some threshold level $l \in [0, m-1]$. In article [3] the M/G/1/m queue which feature consists that after *n* services in a row the forced breaks in work of system occur is studied. Necessity of such pause can be caused by technological or other reasons. For example, the technical device which consistently carries out homogeneous operations, at designing can be calculated on the limited number of operations which are carried out without interruption between operations.

Other modifications of the M/M/1/m queueing system connected with blocking of an input flow and the announcement of a break in work of system after *n* services in a row are offered below. Blocking of an input flow continued from the start of the second service in a row to the moment of free system. We consider two variants of the statement of the problem for the M/M/1/m system with such blocking of an input flow: 1) breaks in work are impossible; 2) after *n* services in a row a break in work of system comes, during which the input flow also is blocked. We also examine another formulation of the first problem when blocking of an input flow continues from the start of the third service in a row.

2. Queueing system with blocking of an input flow. We study single-server queue for which the number of places in queue cannot exceed number *m*. Let customers arrive to system on one. We assume also, that time intervals between the moments of arrival of customers and service times are independent random variables exponentially distributed with parameter λ and μ respectively.

The system accepts customers when it is free and during the first service after the state of idle time of system, if the queue length does not exceed the number *m*.

We introduce double numbering of states of queueing system: s_{kl} designates a state *k* of a level *l*. Here *k* $\left(k = \overline{0, m+1}\right)$ is a number of the customers in system; values of *l* $\left(l = \overline{1, m+1}\right)$ correspond to states of system when the *l*-th customer is served after the period of idle time of system; s_{01} designates a state of free system.

Let us denote by $p_{kl}(t)$ probability of stay of system at time *t* in a state s_{kl}. Evidently (see, for example, [4, p.69; 5, p.61]), that following limits exist

$$p_{kl} = \lim_{t \to \infty} p_{kl}(t) \qquad \left(k = \overline{0, m+1}; \quad l = \overline{1, m+1}\right).$$

Using graph of states of system, we shall write the equations for finding of steady-state probabilities p_{kl}:

$$\mu\sum_{k=1}^{m+1} p_{1k} - \lambda p_{01} = 0; \qquad \lambda p_{k-1,1} - (\lambda+\mu)p_{k1} = 0 \quad \left(k = \overline{1,\, m}\right);$$
$$\lambda p_{m1} - \mu p_{m+1,1} = 0; \tag{1}$$
$$\mu(p_{k+1,l} - p_{k,l+1}) = 0 \qquad \left(l = \overline{1,\, m}; \quad k = \overline{1,\, m-l+1}\right).$$

By means of the equations (1) we can express all probabilities corresponding to states of levels $l \geq 2$ by probabilities of states of the first level p_{k1}:

$$p_{kl} = p_{k+l-1,1} \qquad \left(l = \overline{2,\, m+1}; \quad k = \overline{1,\, m-l+2}\right). \tag{2}$$

We can write the system for probabilities p_{k1} together with the normalization condition in the form of

$$p_{01} = \beta\sum_{k=1}^{m+1} p_{k1}; \qquad p_{k-1,1} = (1+\beta)p_{k1} \quad \left(k = \overline{1,\, m}\right);$$
$$p_{m1} = \beta p_{m+1,1}; \qquad p_{01} + \sum_{k=1}^{m+1} k p_{k1} = 1, \tag{3}$$

where $\beta = \mu/\lambda$. Using equations (3) we can express all the probabilities p_{k1} by $p_{m+1,1}$

$$p_{k1} = \beta(1+\beta)^{m-k} p_{m+1,1} \qquad \left(k = \overline{0,\, m}\right). \tag{4}$$

The probability $p_{m+1,1}$ we find after substitution of the right parts of relations (4) in the normalization condition

$$p_{m+1,1} = \frac{\beta}{\left(\beta^2+\beta+1\right)(1+\beta)^m - 1}. \tag{5}$$

Equalities (2), (4), (5) completely determine ergodic distribution of probabilities of states s_{kl}.

Using this distribution, we can find ergodic distribution of the queue length in the system. Let us denote by π_k steady-state probability that the queue length equals k $\left(k = \overline{0,\, m}\right)$. Then

$$\pi_0 = \sum_{k=0}^{m+1} p_{k1} = (1+\beta)^{m+1} p_{m+1,1} = \frac{\beta(1+\beta)^{m+1}}{\left(\beta^2+\beta+1\right)(1+\beta)^m - 1};$$
$$\pi_k = \sum_{s=k+1}^{m+1} p_{s1} = (1+\beta)^{m-k} p_{m+1,1} = \frac{\beta(1+\beta)^{m-k}}{\left(\beta^2+\beta+1\right)(1+\beta)^m - 1} \qquad \left(k = \overline{1,\, m}\right). \tag{6}$$

Average length of queue we define as the mathematical expectation of discrete random variable:

$$\overline{r} = \sum_{k=1}^{m} k\pi_k = \frac{(1+\beta)^{m+1} - (m+1)\beta - 1}{\beta\left(\left(\beta^2+\beta+1\right)(1+\beta)^m - 1\right)}. \tag{7}$$

Steady-state probability of service of customer, arrived on an input of system, we calculate as the sum of the probabilities of those states, when input flow is not blocked

$$P_{serv} = \sum_{k=0}^{m} p_{k1} = \frac{\beta\left((1+\beta)^{m+1} - 1\right)}{\left(\beta^2+\beta+1\right)(1+\beta)^m - 1}. \tag{8}$$

When we pass to a limit as $m \to \infty$ in relations (4) – (8), we receive ergodic distributions $\{p_{kl}\}$ and $\{\pi_k\}$ and corresponding formulas for the average queue length and probability of service for the M/M/1/∞ system with blocking of an input flow

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$$p_{k1} = \frac{\beta^2}{\left(\beta^2+\beta+1\right)(1+\beta)^k} \qquad (k = 0,\ 1,\ 2,\ldots);$$

$$p_{kl} = p_{k+l-1,1} \quad (k = 1,\ 2,\ldots;\ \ l = 2,3,\ldots);$$

$$\pi_0 = \frac{\beta(1+\beta)}{\beta^2+\beta+1}; \qquad \pi_k = \frac{\beta}{\left(\beta^2+\beta+1\right)(1+\beta)^k} \qquad (k = 1,\ 2,\ldots);$$

$$\overline{r} = \frac{1+\beta}{\beta\left(\beta^2+\beta+1\right)}; \qquad P_{serv} = \frac{\beta(1+\beta)}{\beta^2+\beta+1}.$$

We note that the ergodic process for the M/M/1/∞ system with blocking of an input flow exists for all values of β, unlike the M/M/1/∞ ordinary system, for which it exists only as $\beta > 1$.

3. Queueing system with blocking of an input flow and the breaks in work after *n* services in a row. In the previous problem formulation, we make only one change: after *n* services in a row a break in work of system comes. Time break is exponentially distributed with parameter γ.

Customers who at the time of the announcement of a break are in queue, remain in system and wait on renewal of service. During the interruption new arrivals receiving failure even with the availability of places in the queue. Blocking of an input flow is carried out by the same conditions as in the previous problem, so the problem makes sense only when $m \geq n-1$. Otherwise, the number of customers served in a row is less than *n*.

We keep the above enumeration of states, in addition having designated by $s_{k,n+1}$ $\left(k = \overline{0,\ m-n+1}\right)$ the system states during a forced break. Therefore, the state $s_{k,n+1}$ means presence in system during a break of *k* customers expecting on renewal of service.

First, we assume that $m \geq n,\ \ n \geq 2$, and write the system of equations for the steady-state probabilities p_{kl}:

$$\begin{gathered}
\mu\sum_{k=1}^{n-1} p_{1k} + \gamma p_{0,n+1} - \lambda p_{01} = 0;\\
\lambda p_{k-1,1} + \gamma p_{k,n+1} - (\lambda+\mu)p_{k1} = 0 \quad \left(k = \overline{1,\ m-n+1}\right);\\
\lambda p_{k-1,1} - (\lambda+\mu)p_{k1} = 0 \quad \left(k = \overline{m-n+2,\ m}\right); \qquad \lambda p_{m1} - \mu p_{m+1,1} = 0;\\
\mu(p_{k+1,l} - p_{k,l+1}) = 0 \quad \left(l = \overline{1,\ n-1};\ \ k = \overline{1,\ m-l+1}\right);\\
\mu p_{k+1,n} - \gamma p_{k,n+1} = 0 \quad \left(k = \overline{0,\ m-n+1}\right).
\end{gathered} \tag{9}$$

Equations (9) allow to express the probabilities of states of higher levels $(l \geq 2)$ by the probabilities of states of the first level p_{k1}

$$\begin{gathered}
p_{kl} = p_{k+l-1,1} \quad \left(l = \overline{2,\ n};\ \ k = \overline{1,\ m-l+2}\right);\\
p_{k,n+1} = \delta p_{k+n,1} \quad \left(k = \overline{0,\ m-n+1}\right),
\end{gathered} \tag{10}$$

where $\delta = \mu/\gamma$. The system of equations for the probabilities p_{k1} acquires a form

$$p_{01} = \beta\sum_{k=1}^{n} p_{k1}; \qquad p_{k1} = p_{k-1,1} + \beta\left(p_{k+n,1} - p_{k1}\right) \quad \left(k = \overline{1,\ m-n+1}\right); \tag{11}$$

$$p_{k-1,1} = (1+\beta)p_{k1} \quad \left(k = \overline{m-n+2,\ m}\right); \quad p_{m1} = \beta p_{m+1,1}; \tag{12}$$

$$p_{01} + \sum_{k=1}^{n-1} k p_{k1} + (n+\delta)\sum_{k=n}^{m+1} p_{k1} = 1, \tag{13}$$

where, as above, $\beta = \mu/\lambda$, and (13) is a normalization condition.

Using equations (12) we can express the probabilities p_{k1} $\left(k = \overline{m-n+1,\ m}\right)$ by $p_{m+1,1}$

$$p_{k1} = \beta(1+\beta)^{m-k} p_{m+1,1} = 0 \qquad \left(k = \overline{m-n+1,\ m}\right). \tag{14}$$

From relations (11) we consistently obtain

$$p_{k1} = \beta\sum_{s=1}^{n} p_{k+s,1} \qquad \left(k = \overline{0,\ m-n}\right); \tag{15}$$

Using equalities (14), by means of relations (15), beginning from $p_{m-n,1}$ and further following on decrease of an index *k*, we can consistently express the probabilities p_{k1} $\left(k = \overline{0,\ m-n}\right)$ by $p_{m+1,1}$. The normalization condition (13) serves to calculate $p_{m+1,1}$. So, for all values of *m* and *n*, only if $m \ge n$, we have an algorithm to determine the ergodic distribution p_{kl} $\left(k = \overline{0,\ m+1};\ \ l = \overline{1,\ n+1}\right)$.

Below we examine more in detail such relations between *m* and *n*, if we can find the ergodic distribution in explicit form.

3.1. Case, when $n \le m \le 2n$. From (15) and (14) we have:

$$p_{m-n,1} = \beta\sum_{k=m-n+1}^{m} p_{k1} = \beta^2(1+\beta)^m p_{m+1,1}\sum_{k=m-n+1}^{m}\left(\frac{1}{1+\beta}\right)^k = \beta\left((1+\beta)^n - 1\right)p_{m+1,1};$$

$$p_{m-n-1,1} = \beta\sum_{k=m-n}^{m-1} p_{k1} = \beta p_{m-n,1} + \beta\sum_{k=m-n+1}^{m-1} p_{k1} = \left(\beta^2\left((1+\beta)^n - 1\right) + \beta(1+\beta)\left((1+\beta)^{n-1} - 1\right)\right)p_{m+1,1};$$

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$$p_{m-n-k,1} = \left(\beta^2(1+\beta)^{k-1}\sum_{s=0}^{k-1}\left((1+\beta)^{n-s} - 1\right) + \beta(1+\beta)^k\left((1+\beta)^{n-k} - 1\right)\right)p_{m+1,1} \qquad \left(k = \overline{1,\ m-n}\right).$$

After calculating finite sums and replacement indices we get finally

$$p_{k1} = \beta(1+\beta)^{m-n-k-1}\left((1+\beta)^{n+1} - (m-n-k+1)\beta - 1\right)p_{m+1,1} \qquad \left(k = \overline{0,\ m-n}\right). \tag{16}$$

Formulas (16) are true only for $n \le m \le 2n$. If $m > 2n$, then $m-n > n$ and calculating p_{k1} for $k < m-2n$ by formulas (15) we do not use probabilities p_{k1} $\left(k = \overline{m-n+1,\ m}\right)$, to determine which the relations (14) provide. Therefore, in this case the structure p_{k1} as $k < m-2n$ is different than for $k = \overline{m-2n,\ m-n}$. So if $m > 2n$, then we can use formulas (16) only for $k = \overline{m-2n,\ m-n}$.

After substitution the right parts of of equalities (14) and (16) in the normalization condition (13), which for convenience of calculations we write as

$$p_{01} + \sum_{k=1}^{m-n} k p_{k1} + \sum_{k=m-n+1}^{n-1} k p_{k1} + (n+\delta)\sum_{k=n}^{m+1} p_{k1} = 1,$$

we find the probability $p_{m+1,1}$

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$$p_{m+1,1} = \frac{\beta}{A_{mn}(\beta,\,\delta)},$$

$$A_{mn}(\beta,\,\delta) = (\beta^2+\beta+1)\left((1+\beta)^m - (m-n)\beta(1+\beta)^{m-n-1}\right) + \tag{17}$$

$$+\,\beta(\delta-1)(1+\beta)^{m-n+1} - 1.$$

Thus, the relations (10), (14), (16) and (17) completely determine the ergodic probabilities distribution of the states s_{kl} for a system with blocking of an input flow and interruptions in the work after *n* services in a row if $n \le m \le 2n$.

We can find ergodic distribution of the queue length in the system and average length of queue

$$\pi_0 = \sum_{k=0}^{n} p_{k1} + \delta p_{n1} = \frac{1+\beta}{\beta} p_{01} + \delta p_{n1} =$$

$$= (1+\beta)^{m-n}\left((1+\beta)^{n+1} - (m-n+1-\delta)\beta - 1\right) p_{m+1,1};$$

$$\pi_k = \sum_{s=k+1}^{k+n} p_{s1} + \delta p_{k+n,1} = \frac{1}{\beta} p_{k1} + \delta p_{k+n,1} = (1+\beta)^{m-n-k-1} \times$$

$$\times\left((1+\beta)^{n+1} - (m-n-k+1)\beta + \delta\beta(1+\beta) - 1\right) p_{m+1,1} \qquad \left(k = \overline{1,\,m-n}\right);$$

$$\pi_{m-n+1} = \sum_{k=m-n+2}^{m+1} p_{k1} + \delta p_{m+1,1} = \frac{1}{\beta} p_{m-n+1,1} + \delta p_{m+1,1} = \left((1+\beta)^{n-1} + \delta\right) p_{m+1,1};$$

$$\pi_k = \sum_{s=k+1}^{m+1} p_{s1} = (1+\beta)^{m-k} p_{m+1,1} \qquad \left(k = \overline{m-n+2,\,m}\right);$$

$$\overline{r} = \sum_{k=1}^{m} k\pi_k = \left((1+\beta)^{m+1} - \left((m-n)\beta - 1\right)(1+\beta)^{m-n} + \delta\beta\left((1+\beta)^{m-n+1} - 1\right) - (m+1)\beta - 2\right)\frac{p_{m+1,1}}{\beta^2}.$$

Steady-state probability of service we calculate as the sum of probabilities of states s_{k1} $\left(k = \overline{0,\,m}\right)$

$$P_{serv} = \sum_{k=0}^{m} p_{k1} = \frac{\beta}{A_{mn}(\beta,\delta)}\left((1+\beta)^{m+1} - (m-n+1)\beta(1+\beta)^{m-n} - 1\right).$$

3.2. Case, when $m = n-1$. Unlike the case $m \ge n$ here we have only one state of level $n+1$, this is the state $s_{0,n+1}$, so equations (9) simplifies to the form

$$\mu\sum_{k=1}^{n-1} p_{1k} + \gamma p_{0,n+1} - \lambda p_{01} = 0;$$

$$\lambda p_{k-1,1} - (\lambda+\mu)p_{k1} = 0 \quad \left(k = \overline{1,\,n-1}\right); \qquad \lambda p_{n-1,1} - \mu p_{n1} = 0;$$

$$\mu(p_{k+1,l} - p_{k,l+1}) = 0 \qquad \left(l = \overline{1,\,n-1};\quad k = \overline{1,\,n-l}\right); \qquad \mu p_{1n} - \gamma p_{0,n+1} = 0.$$

As before, we express the probabilities of states of higher levels $(l \ge 2)$ by p_{k1}

$$p_{kl} = p_{k+l-1,1} \qquad \left(l = \overline{2,\,n};\quad k = \overline{1,\,n-l+1}\right); \qquad p_{0,n+1} = \delta p_{n,1} \tag{18}$$

and we write system of equations for the probabilities p_{k1} together with the normalization condition

$$p_{01} = \beta \sum_{k=1}^{n} p_{k1}; \qquad p_{k-1,1} = (1+\beta) p_{k1} \quad \left(k = \overline{1,\, n-1}\right);$$
$$p_{n-1,1} = \beta p_{n1}; \qquad p_{01} + \sum_{k=1}^{n-1} k p_{k1} + (n+\delta) p_{n1} = 1. \tag{19}$$

After solving the system (19), we get

$$p_{k1} = \beta(1+\beta)^{n-k-1} p_{n1} \qquad \left(k = \overline{0,\, n-1}\right);$$
$$p_{n1} = \frac{\beta}{\left(\beta^2+\beta+1\right)(1+\beta)^{n-1} + \delta\beta - 1}. \tag{20}$$

Using the relations (18), (20), we find ergodic distribution of the queue length and steady-state values of average length of queue and probability of service

$$\pi_0 = \sum_{k=0}^{n} p_{k1} + \delta p_{n1} = \frac{\beta\left((1+\beta)^n + \delta\right)}{\left(\beta^2+\beta+1\right)(1+\beta)^{n-1} + \delta\beta - 1};$$
$$\pi_k = \sum_{s=k+1}^{n} p_{s1} = \frac{\beta(1+\beta)^{n-k-1}}{\left(\beta^2+\beta+1\right)(1+\beta)^{n-1} + \delta\beta - 1} \qquad \left(k = \overline{1,\, n-1}\right); \tag{21}$$

$$\overline{r} = \sum_{k=1}^{n-1} k\pi_k = \frac{(1+\beta)^n - n\beta - 1}{\beta\left(\left(\beta^2+\beta+1\right)(1+\beta)^{n-1} + \delta\beta - 1\right)};$$
$$P_{serv} = \sum_{k=0}^{n-1} p_{k1} = \frac{\beta\left((1+\beta)^n - 1\right)}{\left(\beta^2+\beta+1\right)(1+\beta)^{n-1} + \delta\beta - 1}. \tag{22}$$

In the case $m = n-1$ the graph of states of considered queueing system differs from the graph of states of system with blocking of an input flow studied in Section 2 only the presence of additional state $s_{0,n+1}$ which corresponds to the break after n consecutive services. Therefore, putting $n = m+1,\ \delta = 0$ in the formulas (20) – (22), we get the relations (4) – (8).

3.3. Case, when $n = 1$. In this system a break occurs after each service. This case deserves separate consideration because it has the specificity and allows to get the formula for the ergodic probabilities distribution of states for all m $(1 \le m \le \infty)$.

Let us write the equations for a finding of steady-state probabilities p_{kl}:

$$\gamma p_{02} - \lambda p_{01} = 0; \qquad \lambda p_{k-1,1} + \gamma p_{k2} - (\lambda+\mu) p_{k1} = 0 \quad \left(k = \overline{1,\, m}\right);$$
$$\lambda p_{m1} - \mu p_{m+1,1} = 0; \qquad \mu p_{k+1,1} - \gamma p_{k2} = 0 \qquad \left(k = \overline{0,\, m}\right).$$

Expressing the probabilities of states of second level by formulas

$$p_{k2} = \delta p_{k+1,1} \qquad \left(k = \overline{0,\, m}\right),$$

we obtain the relations

$$p_{k1} = \beta p_{k+1,1} \qquad \left(k = \overline{0,\, m}\right),$$

where we have

$$p_{k1} = \beta^{m-k+1} p_{m+1,1} \qquad \left(k = \overline{0,\, m}\right). \tag{23}$$

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Using the normalization condition

$$p_{01} + (1+\delta)\sum_{k=1}^{m+1} p_{k1} = 1$$

and separating cases $\beta \neq 1$ and $\beta = 1$, we find

$$\begin{aligned} p_{m+1,1} &= \frac{\beta - 1}{\beta^{m+2} + \delta\left(\beta^{m+1} - 1\right) - 1}, \qquad \beta \neq 1; \\ p_{k1} &= \frac{1}{1 + (m+1)(1+\delta)} \qquad \left(k = \overline{0,\, m+1}\right), \qquad \beta = 1. \end{aligned} \tag{24}$$

Now we can calculate the ergodic distribution of queue length and steady-state values of average length of queue and probability of service

$$\begin{aligned} \pi_0 &= p_{01} + (1+\delta)p_{11} = \frac{\beta^m(\beta-1)(1+\beta+\delta)}{\beta^{m+2} + \delta\left(\beta^{m+1} - 1\right) - 1}, \qquad \beta \neq 1; \\ \pi_k &= (1+\delta)p_{k+1,1} = \frac{\beta^{m-k}(\beta-1)(1+\delta)}{\beta^{m+2} + \delta\left(\beta^{m+1} - 1\right) - 1} \qquad \left(k = \overline{1,\, m}\right), \qquad \beta \neq 1; \\ \pi_0 &= \frac{2+\delta}{1 + (m+1)(1+\delta)}, \quad \pi_k = \frac{1+\delta}{1 + (m+1)(1+\delta)} \qquad \left(k = \overline{1,\, m}\right), \qquad \beta = 1; \end{aligned} \tag{25}$$

$$\begin{aligned} \overline{r} &= \sum_{k=1}^{m} k\pi_k = \frac{(1+\delta)\left(\beta^{m+1} - (m+1)\beta + m\right)}{(\beta-1)\left(\beta^{m+2} + \delta\left(\beta^{m+1} - 1\right) - 1\right)}, \qquad \beta \neq 1; \\ \overline{r} &= \frac{m(m+1)(1+\delta)}{2\left(1 + (m+1)(1+\delta)\right)}, \qquad \beta = 1; \\ P_{serv} &= \sum_{k=0}^{m} p_{k1} = \frac{\beta\left(\beta^{m+1} - 1\right)}{\beta^{m+2} + \delta\left(\beta^{m+1} - 1\right) - 1}, \qquad \beta \neq 1; \\ P_{serv} &= \frac{m+1}{1 + (m+1)(1+\delta)}, \qquad \beta = 1. \end{aligned} \tag{26}$$

Letting $m \to \infty$ in (23) – (26), we see that the ergodic distribution in the case of absence of restrictions on the length of queue $(m = \infty)$ exists only if $\beta > 1$. Then we have:

$$\begin{aligned} p_{k1} &= \frac{\beta - 1}{\beta^k(\beta + \delta)}, \qquad p_{k2} = \delta p_{k+1,1} \qquad (k = 0,\, 1,\, 2, \ldots); \\ \pi_0 &= \frac{(\beta-1)(1+\beta+\delta)}{\beta(\beta+\delta)}, \qquad \pi_k = \frac{(\beta-1)(1+\delta)}{\beta^{k+1}(\beta+\delta)} \qquad (k = 1,\, 2, \ldots); \\ \overline{r} &= \frac{1+\delta}{(\beta-1)(\beta+\delta)}, \qquad P_{serv} = \frac{\beta}{\beta+\delta}. \end{aligned}$$

We saw in Section 2 that ergodic process for the M/M/1/∞ system with blocking of an input flow exists for any values of β. We can get this system as a result of passage to the limit when $n \to \infty$ and $m \to \infty$ in the system with blocking of an input flow and the breaks in work after *n* services in a row. So, we have bases to predict, that for the M/M/1/∞ system with blocking of an input flow and the breaks in work after *n* services in a

row for sufficiently large values of n existence of an ergodic process is possible even for $\beta \le 1$. If $\beta > 1$, then it certainly exists for all $n \ge 1$.

4. Blocking of an input flow from the beginning of the third service in a row. Let us change the setting of the problem studied in Section 2. Suppose that blocking of an input flow continues from the beginning of the third service in a row to the moment of free system.

Let us write the equations for determining the steady-state probabilities p_{kl}

$$\mu \sum_{k=1}^{m+2} p_{1k} - \lambda p_{01} = 0; \qquad \lambda p_{k-1,1} - (\lambda + \mu) p_{k1} = 0 \quad \left(k = \overline{1,\, m}\right); \tag{27}$$
$$\lambda p_{m1} - \mu p_{m+1,1} = 0;$$

$$\mu p_{21} - (\lambda + \mu) p_{12} = 0; \qquad \lambda p_{k-1,2} + \mu p_{k+1,1} - (\lambda + \mu) p_{k2} = 0 \quad \left(k = \overline{2,\, m}\right); \tag{28}$$
$$\lambda p_{m2} - \mu p_{m+1,2} = 0;$$

$$\mu (p_{k+1,l} - p_{k,l+1}) = 0 \qquad \left(l = \overline{2,\, m+1}; \quad k = \overline{1,\, m+2-l}\right). \tag{29}$$

Using the equations (29) we can express all probabilities corresponding to states of levels $l \ge 3$ by probabilities of states of the second level p_{k2}

$$p_{kl} = p_{k+l-2,2} \qquad \left(l = \overline{3,\, m+2}; \quad k = \overline{1,\, m+3-l}\right).$$

So, the normalization condition for the system of equations (27) – (29) takes form

$$p_{01} + \sum_{k=1}^{m+1} \left(p_{k1} + k p_{k2}\right) = 1. \tag{30}$$

Using the equations (27) we express all the probabilities p_{k1} by $p_{m+1,1}$

$$p_{k1} = \beta (1+\beta)^{m-k} p_{m+1,1} \qquad \left(k = \overline{0,\, m}\right).$$

Rewrite the equations (28) as

$$p_{12} = \frac{\beta}{1+\beta} p_{21}; \qquad p_{m+1,2} = \frac{1}{\beta} p_{m2};$$
$$p_{k2} = \frac{1}{1+\beta} p_{k-1,2} + \frac{\beta}{1+\beta} p_{k+1,1} \quad \left(k = \overline{2,\, m}\right). \tag{31}$$

Recurrence relations (31) allow to express all the probabilities of second level p_{k2} by $p_{m+1,1}$

$$p_{k2} = k\beta^2 (1+\beta)^{m-k-2} p_{m+1,1} \qquad \left(k = \overline{1,\, m-1}\right);$$
$$p_{m2} = \frac{\beta(m\beta+1)}{(1+\beta)^2} p_{m+1,1}; \qquad p_{m+1,2} = \frac{m\beta+1}{(1+\beta)^2} p_{m+1,1}.$$

Finally, by means of normalization condition (30) we find the probability $p_{m+1,1}$

$$p_{m+1,1} = \frac{\beta(1+\beta)^2}{A_m(\beta)},$$
$$A_m(\beta) = \left(\beta^3 + 2\beta^2 + 2\beta + 2\right)(1+\beta)^{m+1} - (1+\beta)^2 - (m\beta+1).$$

Now we can find the ergodic distribution of queue length and steady-state values of average length of queue and probability of service

Transactions of Azerbaijan National Academy of Sciences, Series of Physical-Technical and Mathematical Sciences: Informatics and Control Problems. – 2009. – Vol. XXIX, № 6. – P. 50-58.

$$\pi_0 = p_{01} + \sum_{k=1}^{m+2} p_{1k} = p_{01} + \frac{1+\beta}{\beta} p_{11} = (1+\beta)^{m+1} p_{m+1,1};$$

$$\pi_k = p_{k+1,1} + \sum_{s=k+1}^{m+1} p_{s2} = \left((1+\beta)^2 + k\beta\right)(1+\beta)^{m-k-2} p_{m+1,1} \qquad \left(k = \overline{1,\, m-1}\right);$$

$$\pi_m = p_{m+1,1} + p_{m+1,2} = \frac{(1+\beta)^2 + m\beta + 1}{(1+\beta)^2} p_{m+1,1};$$

$$\bar{r} = \sum_{k=1}^{m} k\pi_k = \frac{1}{\beta A_m(\beta)} \times$$

$$\times\left(\left(\beta^2 + 3\beta + 3\right)(1+\beta)^{m+1} + m\beta^2(m\beta+1) + \left(m^2 - 2m - 2 - \left(m^2+m+1\right)\beta\right)(1+\beta)^2 - (m-1)^2(1+\beta)\right);$$

$$P_{serv} = p_{01} + \sum_{k=1}^{m}(p_{k1} + p_{k2}) =$$

$$= \frac{\beta}{A_m(\beta)}\left(\left(\beta^2 + 2\beta + 2\right)(1+\beta)^{m+1} - (1+\beta)^2 - m\beta - 1\right).$$

When we pass to a limit as $m \to \infty$, we receive ergodic distributions and formulas for the average queue length and probability of service for the corresponding M/M/1/∞ queueing system with blocking of an input flow

$$p_{k1} = \frac{\beta^2}{(1+\beta)^{k-1}\left(\beta^3 + 2\beta^2 + 2\beta + 2\right)} \qquad (k = 0,\, 1,\, 2, \ldots);$$

$$p_{k2} = \frac{k\beta^3}{(1+\beta)^{k+1}\left(\beta^3 + 2\beta^2 + 2\beta + 2\right)} \qquad (k = 1,\, 2, \ldots);$$

$$p_{kl} = p_{k+l-2,2} \qquad (k = 1,\, 2, \ldots;\ \ l = 3,\, 4, \ldots);$$

$$\pi_0 = \frac{\beta(1+\beta)^2}{\beta^3 + 2\beta^2 + 2\beta + 2};$$

$$\pi_k = \frac{\beta(1+\beta)^2 + k\beta^2}{(1+\beta)^{k+1}\left(\beta^3 + 2\beta^2 + 2\beta + 2\right)} \qquad (k = 1,\, 2, \ldots);$$

$$\bar{r} = \frac{\beta^2 + 3\beta + 3}{\beta\left(\beta^3 + 2\beta^2 + 2\beta + 2\right)}; \qquad P_{serv} = \frac{\beta\left(\beta^2 + 2\beta + 2\right)}{\beta^3 + 2\beta^2 + 2\beta + 2}.$$

As well as for queueing system with absence of restrictions on the length of queue considered in Section 2, an ergodic distribution exists here for any values of β.

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